MAXIMUM – MINIMUM PROBLEMS OF TWO VARIABLE FUNCTIONS

RELATIVE MAXIMA-MINIMA

- A function $f(x,y)$ has a **relative maximum** $f(a,b)$ at the point $((a,b), f(a,b))$ if $f(a,b) \geq f(x,y)$ for all (x,y) in some rectangular region about (a,b).
- A function $f(x,y)$ has a **relative minimum** $f(a,b)$ at the point $((a,b), f(a,b))$ if $f(a,b) \leq f(x,y)$ for all (x,y) in some rectangular region about (a,b).

Theorem 1. (Necessary conditions) Suppose that $f(x,y)$ attains a relative maximum or a relative minimum value at the point (a,b) and the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ both exist. Then

$$f_x(a,b) = 0 = f_y(a,b)$$

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Example 1. Find the relative minimum-maximum of

$$f(x,y) = x^2 + y^2, \qquad -\infty \le x, y \le +\infty$$

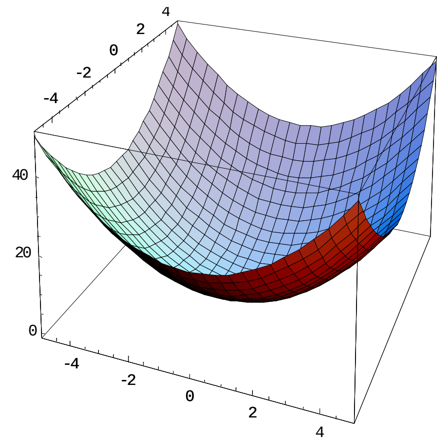
Since the partial derivatives $f_x = \dfrac{\partial f}{\partial x} = 2x$ and $f_y = \dfrac{\partial f}{\partial y} = 2y$ both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

then we **possibly** (since Theorem 1 does not give necessary and sufficient conditions) have a relative minimum-maximum at $(x,y) = (0,0)$.

$$f(x,y) - f(0,0) = x^2 + y^2 - 0 = x^2 + y^2 \ge 0 \Rightarrow f(x,y) \ge f(0,0)$$

Thus we have a minimum at $(x,y) = (0,0)$.

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Plot3D[x^2 + y^2, {x, -5, 5}, {y, -5, 5}, AspectRatio → 1,
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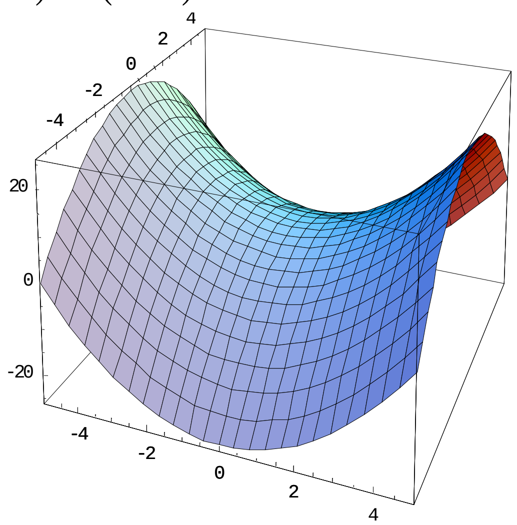
Example 2. Find the relative minimum-maximum of

$$f(x,y) = x^2 - y^2, \qquad -\infty \le x \le +\infty$$

Since the partial derivatives $f_x = \dfrac{\partial f}{\partial x} = 2x$ and $f_y = \dfrac{\partial f}{\partial y} = -2y$ both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

then we **possibly** (since Theorem 1 does not give necessary and sufficient conditions) have a relative minimum-maximum at $(x,y) = (0,0)$.



Note. The theorem gives necessary but not sufficient conditions.

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THE SECOND DERIVATIVE TEST

Suppose that $z = f(x,y)$ has partial derivatives at all points near a point (a,b) and that (a,b) is a critical point of $f(x,y)$ so that

$$f_x(a,b) = 0 = f_y(a,b)$$

Let

$$A = f_{xx}(a,b) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)} \qquad B = f_{xy}(a,b) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)} \qquad C = f_{yy}(a,b) = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)}$$

Rule 1. $f(a,b)$ is a relative maximum if $AC - B^2 > 0$ and $A < 0$.

Rule 2. $f(a,b)$ is a relative minimum if $AC - B^2 > 0$ and $A > 0$.

Rule 3. $((a,b), f(a,b))$ is a saddle point if $AC - B^2 < 0$.

Rule 4. The test gives no information about the type of critical point if $AC - B^2 = 0$.

Note. The values of A,B,C depend upon the point (a,b) and must be determined independently for each critical point.

Example 1. Find the relative minimum-maximum of

$$f(x,y) = x^2 + y^2, \qquad -\infty \le x, y \le +\infty$$

Since the partial derivatives $f_x = \frac{\partial f}{\partial x} = 2x$ and $f_y = \frac{\partial f}{\partial y} = 2y$ both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

$$A = f_{xx}(0,0) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 2 > 0 \quad B = f_{xy}(0,0) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = 0 \quad C = f_{yy}(0,0) = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2$$

$f(0,0) = 0^2 + 0^2 = 0$ is a relative minimum since

$$AC - B^2 = 2 \times 2 - 0^2 = 4 > 0 \text{ and } A = 2 > 0$$

Example 2. Find the relative minimum-maximum of

$$f(x,y)=x^2-y^2, \quad -\infty \le x \le +\infty$$

Since the partial derivatives $f_x=\dfrac{\partial f}{\partial x}=2x$ and $f_y=\dfrac{\partial f}{\partial y}=-2y$ both exist and

$$f_x=f_y=0 \Leftrightarrow (x,y)=(0,0)$$

$$A=f_{xx}(0,0)=\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)}=2>0 \quad B=f_{xy}(0,0)=\left.\frac{\partial^2 f}{\partial y\partial x}\right|_{(0,0)}=0 \quad C=f_{yy}(0,0)=\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)}=-2$$

$\left((0,0),\ f(0,0)=0\right)$ is a saddle point since

$$AC-B^2=2\times(-2)-0^2=-4<0$$

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ABSOLUTE MAXIMUM-MINIMUM

- The **absolute maximum** of the function $f(x,y)$ is a value $f(a,b)$ such that $f(a,b) \geq f(x,y)$ for **all values** of (x,y) in the domain of $f(x,y)$.
- The **absolute maximum** of the function $f(x,y)$ is a value $f(a,b)$ such that $f(a,b) \leq f(x,y)$ for **all values** of (x,y) in the domain of $f(x,y)$.

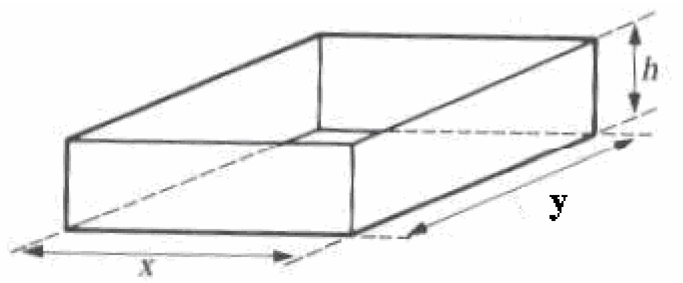
How to find the absolute maximum-minimum of the continuous function $f(x,y)$ on a closed curve R;

1. Find all the critical values of $f(x,y)$ i.e. $f_x(a,b) = 0 = f_y(a,b)$.
2. Find the maximum of the values of $f(x,y)$ on the boundaries of R.
3. Find the values of $f(x,y)$ for the points where there is no partial derivative of $f(x,y)$.
4. Of the above values, the largest is the absolute maximum, and the smallest is the absolute minimum.

Box optimization

Find the dimensions of a rectangular box (with no top) of volume 64in^3, that has the minimum surface area.

Step 1. Understand the problem.



V=volume of the box
E=surface area of the box (no top)
x=the length of the first side
y= the length of the second side
h=the height of the box

$$V = xyh = 64 \qquad \text{and} \qquad E = xy + 2xh + 2yh$$

Step 2. Form a mathematical statement of the problem.

$$h=\frac{64}{xy}$$

$$E(x,y)=xy+2x\frac{64}{xy}+2\mathrm{y}\frac{64}{\mathrm{xy}}=\mathrm{xy}+\frac{128}{\mathrm{y}}+\frac{128}{\mathrm{x}} \qquad 0<x,y$$

We seek the maximum value of $E(x,y)$ for $0<x,y$.

Step 3. Determine the maximum-minimum.

Find the derivative of $E(x,y)$:

$$E_x(x,y)=y-\frac{128}{x^2}=\frac{yx^2-128}{x^2} \text{ and } E_y(x,y)=x-\frac{128}{y^2}=\frac{xy^2-128}{y^2}$$

$$E_y(x,y)=0\Rightarrow xy^2-128=0\Rightarrow x\left(\frac{128}{x^2}\right)^2-128=0\Rightarrow\frac{128^2}{x^3}=128\Rightarrow128=x^3\Rightarrow x=\sqrt[3]{128}=4\sqrt[3]{2}$$

$$y=\frac{128}{x^2}=\frac{128}{\left(4\sqrt[3]{2}\right)^2}=4\sqrt[3]{2}$$

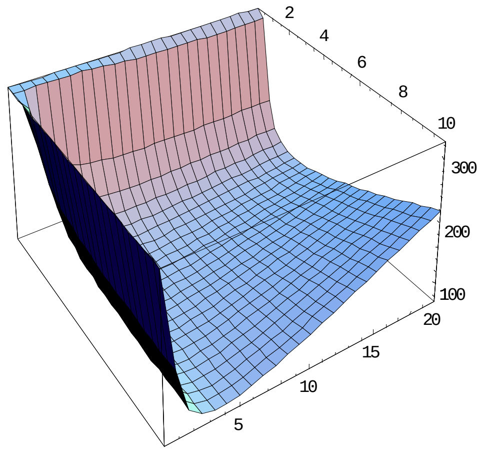
Note also that

$$A = E_{xx}\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \left.\frac{2\times 128}{x^3}\right|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 2 > 0 \quad B = E_{xy}\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \left.\frac{\partial^2 E}{\partial y \partial x}\right|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 1$$

$$C = E_{yy}\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \left.\frac{2\times 128}{y^3}\right|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 2 > 0$$

$f\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \left.\mathrm{xy} + \frac{128}{\mathrm{y}} + \frac{128}{\mathrm{x}}\right|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 48\sqrt[3]{4}$ is a relative minimum since

$AC - B^2 = 2\times 2 - 1^2 = 3 > 0$ and $A = 2 > 0$



Step 4. Answer the question.

Since we have the absolute maximum for $\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right)$, we have that $h = \frac{128}{xy} = \frac{128}{4\sqrt[3]{2} \times 4\sqrt[3]{2}} = 4\sqrt[3]{2}$ and therefore the dimensions of the box with the maximum surface area are $4\sqrt[3]{2}$ inches by $4\sqrt[3]{2}$ inches by $4\sqrt[3]{2}$ inches.

Note. Find the dimensions of a rectangular box (with no top) with surface area of 64in^2, that has the maximum volume.

Least squares method

Find the line $y = ax + b$ that best fits the data points $(x_i, y_i), i = 1,..,n$

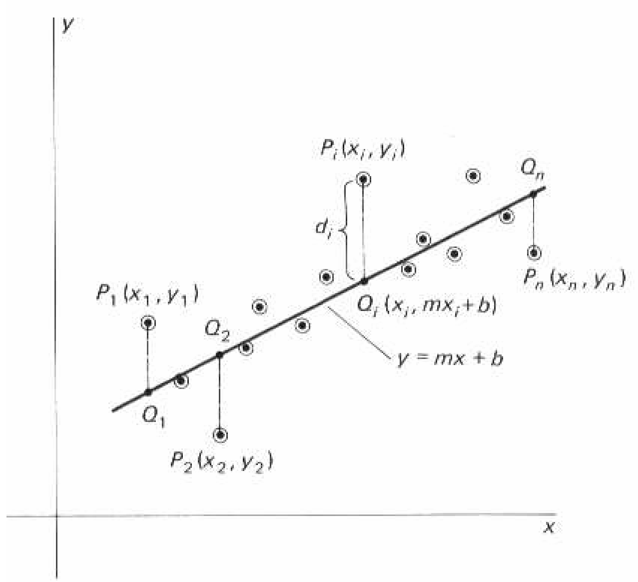
Solution

We define the deviation between the ith point and the line to be :

$$d_i = y_i - (ax_i + b)$$

We need to minimize the function

$$f(a,b) = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



$$f_a(a,b)=\frac{\partial}{\partial a}\left\{\sum_{i=1}^{n}\left(y_i-ax_i-b\right)^2\right\}=\sum_{i=1}^{n}2\left(y_i-ax_i-b\right)\left(-x_i\right)=2a\sum_{i=1}^{n}x_i^2+2b\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}x_iy_i$$

$$f_b(a,b)=\frac{\partial}{\partial b}\left\{\sum_{i=1}^{n}\left(y_i-ax_i-b\right)^2\right\}=\sum_{i=1}^{n}2\left(y_i-ax_i-b\right)\left(-1\right)=2a\sum_{i=1}^{n}x_i+2b\sum_{i=1}^{n}1-2\sum_{i=1}^{n}y_i$$

Solve

$$f_a(a,b)=f_b(a,b)=0$$

or equivalently

$$a\sum_{i=1}^{n}x_i^2+b\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}x_iy_i$$

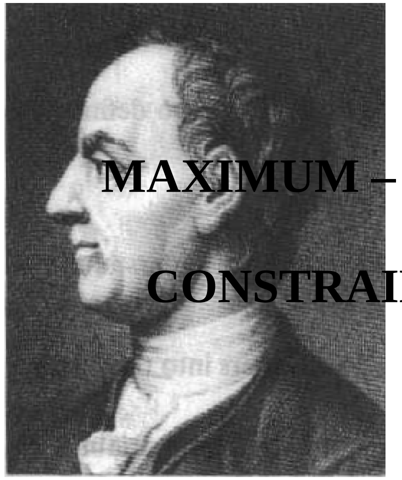
$$a\sum_{i=1}^{n}x_i+bn=\sum_{i=1}^{n}y_i$$

or

$$\begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \begin{pmatrix} n & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix} \Rightarrow$$

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}, b = \frac{-\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right) + \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Note. Use the second derivative test in order to show that we have a minimum for these (a,b).



MAXIMUM – MINIMUM PROBLEMS OF TWO VARIABLE FUNCTIONS

CONSTRAINED OPTIMIZATION PROBLEMS AND LAGRANGE MULTIPLIERS

Minimize-Maximize $Q(x,y)$ subject to constraints $C(x,y)=0$.

Define the Lagrange function as

$$L(x,y,\lambda)=Q(x,y)+\lambda C(x,y)$$

Then all the relative minimum and maximum points of $Q(x,y)$ with x and y constrained to satisfy the equation $C(x,y)=0$ will be among those points (x_0,y_0) for which (x_0,y_0,λ_0) is a maximum or minimum point of $L(x,y,\lambda)$. These points (x_0,y_0,λ_0) will be solutions of the system of simultaneous equations

$$L_x(x,y,\lambda)=0$$

$$L_y(x,y,\lambda)=0$$

$$L_\lambda(x,y,\lambda)=0 \quad \text{(this is just } C(x,y)=0\text{)}$$

We assume that all partial derivatives exist.

Example. Minimize $E = (2x)\times(2y) = 4xy$ subject to constraints $x^2 + y^2 = 20^2$.
Define the Lagrange function as

$$L(x, y, \lambda) = 4xy + \lambda\left(x^2 + y^2 - 20^2\right)$$

Find the points (x_0, y_0, λ_0) that satisfy the system of simultaneous equations

$$L_x(x, y, \lambda) = 4y + 2\lambda x = 0 \quad L_y(x, y, \lambda) = 4x + 2\lambda y = 0$$

$$L_\lambda(x, y, \lambda) = x^2 + y^2 - 20^2 = 0$$

or equivalently

$$y = -\frac{1}{2}\lambda x$$

$$x = -\frac{1}{2}\lambda y \Rightarrow x = -\frac{1}{2}\lambda\left(-\frac{1}{2}\lambda x\right) = \frac{1}{4}\lambda^2 x \Rightarrow \left(1 - \frac{\lambda^2}{4}\right)x = 0 \Rightarrow$$

$$x = 0 \vee \lambda = \pm 2$$

For $x = 0$ we have that $y = 0$ but $0^2 + 0^2 \neq 20^2$. Therefore $x = -y$ for $\lambda = 2$ (or $y = x$ for $\lambda = -2$) and

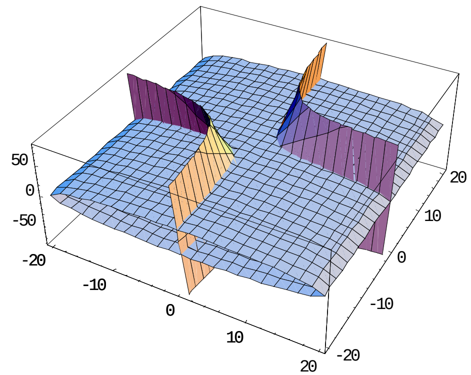
$$x^2 + y^2 - 20^2 = 0 \Rightarrow x^2 + (-x)^2 - 20^2 = 0 \Rightarrow 2x^2 = 20^2 \Rightarrow x = \pm 10\sqrt{2}$$

(respectively $x^2 + y^2 - 20^2 = 0 \Rightarrow x^2 + x^2 - 20^2 = 0 \Rightarrow 2x^2 = 20^2 \Rightarrow x = \pm 10\sqrt{2}$)

and thus

$$y = -x = \mp 10\sqrt{2} \text{ (respectively } y = x = \pm 10\sqrt{2}\text{)}$$

Thus the only candidates for minimum-maximum points of the constrained optimization problem are the pairs $\left(\pm 10\sqrt{2}, \pm 10\sqrt{2}\right)$ and $\left(\pm 10\sqrt{2}, \mp 10\sqrt{2}\right)$. The values of the function $E = (2x)\times(2y) = 4xy$ at these points are respectively $E\left(\pm 10\sqrt{2}, \pm 10\sqrt{2}\right) = 4\left(\pm 10\sqrt{2}\right)\left(\pm 10\sqrt{2}\right) = 800$ and $E\left(\pm 10\sqrt{2}, \mp 10\sqrt{2}\right) = 4\left(\pm 10\sqrt{2}\right)\left(\mp 10\sqrt{2}\right) = -800$. Therefore we have a minimum for the pair $\left(10\sqrt{2}, -10\sqrt{2}\right)$ (and $\left(-10\sqrt{2}, 10\sqrt{2}\right)$) and maximum for the pair $\left(10\sqrt{2}, 10\sqrt{2}\right)$ (and $\left(-10\sqrt{2}, -10\sqrt{2}\right)$).



Exersizes

1. (The cake-pan : optimization) A manufacturing company plans to make microwave-safe cake pans by cutting squares out of the corners of a 12-inch by 16-inch piece of plastic, and then bending the sides up. The seam along each corner will be fused to finish the pan.
The manufacturer wishes to determine what size square should be cut from the corners to make a pan of the greatest possible volume.

2. Find the point on the plane

$$x + 2y + 2z = 4$$

that is closest to the origin.