ARISTOTLE UNIVERSITY OF THESSALONIKI
DEPARTMENT OF MATHEMATICS
MASTER'S PROGRAMME
THEORETICAL INFORMATICS AND SYSTEMS & CONTROL THEORY

Modeling of discrete time auto-regressive systems with given forward and backward behavior

M.Sc. THESIS

MOYSIS S. LAZAROS

Supervisor: Nicholas Karampetakis
Associate Professor, Aristotle University Of Thessaloniki

Thessaloniki, February 2013

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Εγκρίθηκε από την τριμελή εξεταστική επιτροπή την

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Α. Ι. Βαρδουλάκης	Ν. Καραμπετάκης	Ε. Αντωνίου
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Μοντελοποίηση πραγματικών χρονοσειρών μέσω συστημάτων αυτοπαλινδρόμησης

ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

ΜΩΥΣΗΣ Σ. ΛΑΖΑΡΟΣ

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Θεσσαλονίκη, Φεβρουάριος 2013

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Μωυσής Λάζαρος

Πτυχιούχος Μαθηματικός Α.Π.Θ.

Η ολοκλήρωση αυτής της διπλωματικής εργασίας συγχρηματοδοτήθηκε μέσω του Έργου

«Υποτροφίες ΙΚΥ», από πόρους του ΕΠ «Εκπαίδευση και Δια Βίου Μάθηση», του Ευρωπαϊκού Κοινωνικού Ταμείου (ΕΚΤ) του ΕΣΠΑ, 2007-2013.

Τους ευχαριστώ προσωπικά για την οικονομική τους υποστήριξη .

To **IRON MAIDEN**

"But now it seems, I'm just a stranger to myself"

ΠΕΡΙΛΗΨΗ

Στην παρούσα διπλωματική εργασία γίνεται μελέτη των συστημάτων αλγεβρικών εξισώσεων διαφορών που βρίσκονται στη μορφή AR-representation, δηλαδή στη μορφή $A(\sigma)\beta(k)=0$, όπου $A(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$ και $\sigma\beta(k)=\beta(k+1)$. Η εργασία είναι χωρισμένη σε δύο μέρη.

Στο πρώτο μέρος μελετάμε τις λύσεις των συστημάτων αλγεβρικών εξισώσεων διαφορών, οι οποίες χωρίζονται σε δύο μεγάλες κατηγορίες, που συνδέονται με τους πεπερασμένους και τους άπειρους στοιχειώδεις διαιρέτες του συστήματος αντίστοιχα.

Στο δεύτερο μέρος μελετάμε το αντίστροφο πρόβλημα. Έχοντας δεδομένη τη συμπεριφορά ενός συστήματος, να κατασκευάσουμε τον πίνακα A(σ) ώστε να ικανοποιεί τη δοσμένη συμπεριφορά. Δίνουμε ένα θεώρημα που συνδέει τη backward συμπεριφορά ενός συστήματος με την forward συμπεριφορά του δυικού του. Δίνουμε επιπλέον δύο μεθόδους κατασκευής ενός συστήματος με δεδομένη forward και backward συμεπριφορά.

ΛΕΞΕΙΣ ΚΛΕΙΔΙΑ

πολυωνυμικοί πίνακες, κατασκευή συστήματος, στοιχειώδεις διαιρέτες, εξισώσεις διαφορών, γραμμικό σύστημα, ζεύγη Jordan, χρονοσειρές, μοντελοποίηση συστήματος.

ABSTRACT

In the present Thesis we study the forward and backward-impulsive behavior of systems of algebraic difference equations in the form of AR-representations, that is $A(\sigma)\beta(k)=0$, where $A(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$ and $\sigma\beta(k)=\beta(k+1)$. This paper consists of two parts.

In the first part we study the forward and backward solutions of AR-representations, which are depended upon the finite and infinite elementary divisors of the matrix A(σ).

In the second part we study the inverse problem. That is, given a given behavior, how to construct a system that satisfies it. First, we give a theorem connecting the backward behavior of a system to the forward behavior of its dual system. We also present two methods of constructing a system with a given forward and backward behavior.

KEY WORDS

polynomial matrix, forward behavior, backward behavior, Jordan pairs, elementary divisors, solution space, linear system, difference equations, system modeling, time series.

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CHAPTER 1

PRELIMINARY RESULTS

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}[s]$ be the Euclidean ring of polynomials with coefficients from $\mathbb{R}$. The field of all $m\times n$ matrices with elements from $\mathbb{R}[s]$ is denoted by $\mathbb{R}[s]^{m\times n}$. A matrix whose elements are polynomials is called a polynomial matrix and can be expanded as follows:

$$A(s)=A_q s^q+A_{q-1}s^{q-1}+\ldots+A_0\in\mathbb{R}^{m\times n}\ ;A_i\in\mathbb{R}^{m\times n},A_q\neq 0$$

The number $q\in\mathbb{Z}^+$ is the highest degree occurring among the degrees of all the polynomial elements of A(σ).

Example 1: The matrix

$$A(s)=\begin{pmatrix} s^3+2s^2+1 & s^2 \\ s^3-1 & s^2+3 \end{pmatrix}$$

can be expanded as

$$\begin{pmatrix} s^3+2s^2+1 & s^2 \\ s^3-1 & s^2+3 \end{pmatrix}=s^3\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{A_3}+s^2\underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}_{A_2}+\underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}}_{A_0}$$

□

Definition 2 [Vardulakis 1991]: The *degree* of a polynomial matrix T(s), denoted by degT(s) is defined as the maximum degree among the degrees of all its maximum order non-zero minors. □

Definition 3 [Vardulakis 1991]: A matrix $T(s)\in\mathbb{R}[s]^{r\times r}$ is called unimodular, if there exists a $\hat{T}(s)\in\mathbb{R}[s]^{r\times r}$ such that $T(s)\hat{T}(s)=I_r$, or equivalently if detT(σ)= $c\in\mathbb{R}$. □

Definition 4 [Vardulakis 1991]: A matrix T(s) is called proper rational matrix if all its elements are proper rational functions or are equivalently in the form $\dfrac{n(s)}{d(s)}$, $n(s),d(s)\in\mathbb{R}[s]$ with $\deg\left(d(s)\right)\geq\deg\left(n(s)\right)$. We can denote this by $T(s)\in\mathbb{R}_{pr}^{r\times r}(s)$. □

Definition 5 [Vardulakis 1991]: A matrix $T(s) \in \mathbb{R}^{r \times r}_{proper}(s)$ is called $\mathbb{R}_{pr}(s)-$unimodular or *biproper* rational matrix if there exists $\tilde{T}(s) \in \mathbb{R}^{r \times r}_{proper}(s)$ such that $T(s)\tilde{T}(s) = I_r$ or equivalently iff deg[detT(s)]=0.

□

Defition 6 [Vardulakis 1991]: Elementary row and column operations on any polynomial matrix $T(s) \in \mathbb{R}[s]^{r \times m}$ are defined as:

- The interchange any two rows or columns of T(s).
- Multiplication any row or column of T(s) by a non-zero constant from $\mathbb{R}$.
- Addition to any row(column) of T(s) another row(column) multiplied by any polynomial w(s)

□

We are going to study the solution space of systems of difference equations that are in the form of an (Auto-Regressive) AR-representation, that is

$$A(\sigma)\beta(k) = 0 \qquad \textbf{(1)}$$

where $k \in [0, N-q]$, or equivalently

$$A_q\beta(k+q) + A_{q-1}\beta(k+q-1) + \ldots + A_0\beta(k) = 0$$

and

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0\sigma^0 \in \mathbb{R}^{r \times r}[\sigma] \qquad \textbf{(2)}$$

is a regular polynomial matrix, i.e. $\det[A(\sigma)] \neq 0$ for almost every σ, $\beta(k) \in \mathbb{R}^{r \times 1}$, $k \in [0, N]$ and σ denotes the forward shift operator $\sigma\beta(k) = \beta(k+1)$. The number q is also called the *lag* of the system and it denotes the maximum number of time shifts.

Example 7: The difference equation $x(k+2) - x(k+1) - x(k) = 0$ which describes the famous Fibonacci sequence can be written in AR form as

$$\sigma^2 x(k) - \sigma x(k) - x(k) = 0 \rightarrow .$$

$$\left(\sigma^2 - \sigma - 1\right) x(k) = 0 \rightarrow$$

$$\mathrm{A}(\sigma)x(k) = 0$$

□

We define the behavior B of (1) as B={β(k):[0,N] $\rightarrow \mathbb{R}^{r\times 1}$ | (1) is satisfied $\forall k \in [0, N]$ Notice that we are interested in finding the behavior of (1) over a specified finite time interval $k \in [0, N]$, though N is considered to be large enough.

Definition 8 [Vardulakis 1991]: Let A(σ) be an $r \times r$ regular polynomial matrix. There exist unimodular matrices $U_L(\sigma) \in \mathbb{R}[\sigma]^{r\times r}, U_R(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ such that

$$
\begin{aligned}
&S^{\mathbb{C}}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = \\
&= blockdiag\left[1,...,1, f_z(\sigma), f_{z+1}(\sigma),..., f_r(\sigma)\right]
\end{aligned}
$$

with $1 \le z \le r$ and $f_i(\sigma) / f_{i+1}(\sigma)$ i=z,z+1,…,r. $S^{\mathbb{C}}_{A(\sigma)}(\sigma)$ is called the Smith form of A(σ).

Proof: Among the entries of A(σ), we find a non-zero one, which is a polynomial of the lowest degree with respect to σ and by interchanging rows and columns we move it to position (1,1). Denote this entry by $\bar{a}_{11}(\sigma)$. Assume at the beginning that all the entries of the matrix A(σ) are divisible without remainder by $\bar{a}_{11}(\sigma)$. Dividing the entries $\bar{a}_{i1}(\sigma)$ of the first column and the first row $\bar{a}_{1j}(\sigma)$ by $\bar{a}_{11}(\sigma)$ we obtain

$$\bar{a}_{i1}(\sigma) = \bar{a}_{11}(\sigma)q_{i1}(\sigma) \qquad \text{i=2,…,r}$$

$$\bar{a}_{1j}(\sigma) = \bar{a}_{11}(\sigma)q_{1j}(\sigma) \qquad \text{j=2,…,r}$$

where $q_{i1}(\sigma)$ and $q_{1j}(\sigma)$ are the quotients of the divisions.

Subtracting from the i-th row (i = 2,3,…,r) the first row multiplied by $q_{i1}(\sigma)$ and, respectively from the j-th column (j = 2,3,…,m) the first column multiplied by $q_{1j}(\sigma)$, we obtain a matrix of the form

$$
\begin{pmatrix}
\bar{a}_{11}(\sigma) & 0 & \cdots & 0 \\
0 & \bar{a}_{22}(\sigma) & \cdots & \bar{a}_{2r}(\sigma) \\
\vdots & \vdots & \ddots & \vdots \\
0 & \bar{a}_{r2}(\sigma) & \cdots & \bar{a}_{rr}(\sigma)
\end{pmatrix}
$$

If the coefficient by the highest power of s of polynomial $\bar{a}_{11}(\sigma)$ is not equal to 1, then to accomplish this we multiply the first row (or column) by the reciprocal of this coefficient.

Assume next that not all entries of the matrix A(s) are divisible without remainder by $\bar{a}_{11}(\sigma)$ and that such entries are placed in the first row and the first column. Dividing the entries of the first row and the first column by $\bar{a}_{11}(\sigma)$ we obtain

$$\bar{a}_{1i}(\sigma) = \bar{a}_{11}(\sigma)q_{1i}(\sigma) + r_{1i}(\sigma) \qquad i=2,\dots,r$$

$$\bar{a}_{j1}(\sigma) = \bar{a}_{11}(\sigma)q_{j1}(\sigma) + r_{j1}(\sigma) \qquad i=2,\dots,r$$

where $q_{1i}(\sigma), q_{j1}(\sigma)$ are the quotients and $r_{1i}(\sigma), r_{j1}(\sigma)$ are the remainders of the divisions. Subtracting from the j-th row (i-th column) the first row (column) multiplied by $q_{j1}(\sigma)$ (by $q_{1i}(\sigma)$), we replace the entry $\bar{a}_{j1}(\sigma)$ ($\bar{a}_{1i}(\sigma)$) by the remainder $r_{j1}(\sigma)$ ($r_{1i}(\sigma)$).Next, among these remainders we find a polynomial of the lowest degree with respect to sand interchanging rows and columns, we move it to the position (1,1). We denote this polynomial by $\bar{r}_{11}(\sigma)$. If not all entries of the first row and the first column are divisible without remainder by $\bar{r}_{11}(\sigma)$, then we repeat this procedure taking the polynomial $\bar{r}_{11}(\sigma)$ instead of the polynomial $\bar{a}_{11}(\sigma)$. The degree of the polynomial $\bar{r}_{11}(\sigma)$ is lower than the degree of $\bar{a}_{11}(\sigma)$.After a finite number of steps, we obtain in the position (1,1) a polynomial that divides without remainder all the entries of the first row and the first column. If the entry $\bar{a}_{ik}(\sigma)$ is not divisible by $\bar{a}_{11}(\sigma)$ then by adding the i-th row (or k-th column) to the first row (the first column), we reduce this case to the previous one. Repeating this procedure, we finally obtain in the position (1,1) a polynomial that divides without remainder all the entries of the matrix. Further we proceed in the same way as in the first case, when all the entries of the matrix are divisible without remainder by $\bar{a}_{11}(\sigma)$.

If not all entries $\bar{a}_{ij}(\sigma)$ (i=2,…,r j=2,…,r) of the previous matrix are equal to zero, we find a non-zero entry of the lowest degree among them and by elementary row and column operations we bring it to position (2,2). Proceeding further as above we obtain a matrix of the form

$$\begin{pmatrix} \bar{a}_{11}(\sigma) & 0 & 0 & \cdots & 0 \\ 0 & \bar{a}_{22}(\sigma) & 0 & \cdots & 0 \\ 0 & 0 & \bar{a}_{33}(\sigma) & \cdots & \bar{a}_{3r}(\sigma) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \bar{a}_{r3}(\sigma) & \cdots & \bar{a}_{rr}(\sigma) \end{pmatrix}$$

where $\bar{a}_{22}(\sigma)$ is divisible without remainder by $\bar{a}_{11}(\sigma)$ and all elements $\bar{a}_{ij}(\sigma)$ (i = 3,4,…,r; j = 3,4,…,r) are divisible without remainder by $\bar{a}_{22}(\sigma)$. Continuing this procedure, we obtain a matrix of the Smith canonical form. □

Polynomials $f_i(\sigma)$ are called the *invariant polynomials* of A(σ). The zeros $s_i \in \mathbb{C}$ of $f_j(s) \in \mathbb{R}[s]$, j=z,z+1,…,r are called *finite zeros* of A(s). Assume that the partial multiplicities of each zero $s_i \in \mathbb{C}$, $i \in k$ are $0 \le n_{i,z} \le n_{i,z+1} \le ... \le n_{i,r}$ i.e.

$$f_j(s) = (s-s_i)^{n_{i,j}}\, \hat{f}_j(s),\ \text{j=z,z+1,…,r}\ ;\ \hat{f}_j(s_i) \neq 0$$

The terms $(s-s_i)^{n_{i,j}}$ are called *finite elementary divisors* of A(s) at $s = s_i$. We also denote by n the sum of the degrees of the finite elementary divisors of A(s), i.e.

$$n := \deg\left[\prod_{j=z}^{r} f_j(s)\right] = \sum_{i=1}^{k}\sum_{j=z}^{r} n_{i,j}$$

Similarly, we can find $U_L(\sigma) \in \mathbb{R}^{r\times r}, U_R(\sigma) \in \mathbb{R}^{r\times r}$ having no poles and zeros at $\sigma = \lambda_0$ such that

$$S_{A(\sigma)}^{\lambda_0}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = blockdiag\left[1,1,...,1,(\sigma-\lambda_0)^{n_z},(\sigma-\lambda_0)^{n_{z+1}},...,(\sigma-\lambda_0)^{n_r}\right]$$

$S_{A(\sigma)}^{\lambda_0}(\sigma)$ is called the Smith form at the local point $\sigma = \lambda_0$.

Lemma 9 [Vardulakis 1991]: The previous algorithm to compute the Smith form of a polynomial matrix A(s) can be summarized in the following steps. By doing biproper row and column operations, follow the next:

Step 1: Move the element with the lowest degree to position [1,1].

Step 2: Reduce all elements of the first column to zero.

Step 3: Reduce all elements of the first row to zero.

Step 4: In case non zero elements appeared on the first column, go back to Step 2.

Step 5: In case the element [1,1] does not divide all the elements of matrix A(s), then go back to Step 1.

Step 6: We will end up with a matrix of the form $\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & B_1(s) \end{pmatrix}$. Go back to Step 1 and do the algorithm again with $B_1(s)$ as the initial matrix.

Example 10: Consider the matrix

$$P(\sigma)=\begin{pmatrix}\sigma & \sigma^2\\ 1 & -1+\sigma\end{pmatrix}$$

There exist $U_L(\sigma)\in\mathbb{R}[\sigma]^{2\times 2}, U_R(\sigma)\in\mathbb{R}[\sigma]^{2\times 2}$ such that

$$\underbrace{\begin{pmatrix}1 & 0\\ 0 & \sigma\end{pmatrix}}_{S^{\mathbb{C}}_{A(\sigma)}(\sigma)}=\underbrace{\begin{pmatrix}0 & -1\\ 1 & -\sigma\end{pmatrix}}_{U_L}\underbrace{\begin{pmatrix}\sigma & \sigma^2\\ 1 & -1+\sigma\end{pmatrix}}_{P}\underbrace{\begin{pmatrix}-1 & 1-\sigma\\ 0 & 1\end{pmatrix}}_{U_R}$$

We have defined the finite elementary divisors with the help of the Smith form of A(σ). Now we will define the infinite elementary divisors of A(σ). To do so, we first need to introduce the Smith form at infinity and the dual polynomial matrix of A(σ).

Definition 11 [Vardulakis 1991]: Let A(σ) be an $r\times r$ polynomial matrix. Then there exist biproper matrices $U_L(\sigma)\in\mathbb{R}^{r\times r}_{pr}(s), U_R(\sigma)\in\mathbb{R}^{r\times r}_{pr}(s)$ such that

$$\mathrm{U}_L(\sigma)\mathrm{A}(\sigma)\mathrm{U}_R(\sigma)=S^{\infty}_{\mathrm{A}(\sigma)}(\sigma)=blockdiag\left(\sigma^{q_1},\ldots,\sigma^{q_k},\frac{1}{\sigma^{\tilde{q}_{k+1}}},\ldots,\frac{1}{\sigma^{\tilde{q}_r}}\right)$$

where

$$q_1\geq q_2\geq\ldots\geq q_k\geq 0$$

$$\tilde{q}_r\geq\tilde{q}_{r-1}\geq\ldots\tilde{q}_{k+1}\geq 0$$

$S^{\infty}_{\mathrm{A}(\sigma)}(\sigma)$ is called the Smith form of A(σ) at infinity. The first k terms $q_1,...,q_k$ are the poles and the latter (r-k) terms $\tilde{q}_{k+1},...,\tilde{q}_r$ the zeros at $\sigma=\infty$ of A(σ). □

It is proved in [Vardulakis, 1991] that $q_1=q$.

There is a simple and pretty straightforward way of finding the Smith form at Infinity of a matrix. It is handy because it avoids performing row and column operations on a matrix A(s).

Definition 12 [Vardulakis 1991, Jones 1998]: Let $g(s)=\dfrac{n(s)}{d(s)}\in\mathbb{R}(s)$ where $\mathrm{n(s),d(s)}\in\mathbb{R}[s]$, $d(s)\neq 0$ and define the mapping $\delta_\infty(\cdot):\mathbb{R}(s)\to\mathbb{Z}\cup\{+\infty\}$ such that

This mapping is defined as a discrete valuation of $\mathbb{R}(s)$. □

Lemma 13 [Vardulakis 1991, Jones 1998]: Let $A(s)\in\mathbb{R}^{m\times n}$, rank(A(s))=r and denote $\xi_i(A)\in\mathbb{Z}$ as the least $\delta_\infty(\cdot)$ of all minors of A(s) of order i, where $\xi_0(A)=0$. Then define

$$
\begin{aligned}
q_1 &= \xi_0(A)-\xi_1(A) = -\xi_1(A)\\
q_2 &= \xi_1(A)-\xi_2(A)\\
&\dots\\
q_r &= \xi_{r-1}(A)-\xi_r(A)
\end{aligned}
$$

The Smith form at infinity of A(s) is given by

$$S^{\infty}_{\mathrm{A}(\sigma)}(\sigma) = blockdiag\left(s^{q_1} \quad s^{q_2} \quad \dots \quad s^{q_r} \quad 0_{m-r\times n-r}\right)$$

□

Example 14: Consider the matrix $A=\begin{pmatrix} s^2 & s+1 \\ 1 & s^2-1\end{pmatrix}$. Define

$$
\begin{aligned}
\xi_0(A) &= 0\\
\xi_1(A) &= \min\{-2,-1,0\} = -2\\
\xi_2(A) &= \min\{-4\} = -4
\end{aligned}
$$

so we have that $q_1 = 0-(-2)=2, q_2=-2-(-4)=2$ so

$$S^{\infty}_{\mathrm{A}(\sigma)}(\sigma) = \begin{pmatrix} s^2 & 0 \\ 0 & s^2\end{pmatrix}$$

□

Definition 15 [Vardulakis 1991]: The dual polynomial matrix of $\mathrm{A}(\sigma)\in\mathbb{R}^{r\times r}$ is defined as

$$\tilde{\mathrm{A}}(\sigma) := \sigma^q A(\frac{1}{\sigma}) = A_0\sigma^q + A_1\sigma^{q-1}+\dots+A_q \qquad \textbf{(3)}$$

□

Let $\tilde{U}_L(\sigma)\in\mathbb{R}[\sigma]^{r\times r}, \tilde{U}_R(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$ be rational matrices having no poles and zeros at σ=0 such that

$$S^{0}_{\tilde{\mathrm{A}}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{\mathrm{A}}(\sigma)\tilde{U}_R(\sigma) = blockdiag\left[\sigma^{\mu_1},\sigma^{\mu_2},\dots,\sigma^{\mu_r}\right]$$

$S^0_{\tilde{A}(\sigma)}(\sigma)$ is the Smith form of $\tilde{A}(\sigma)$ at zero. The terms σ^{μ_j} are the finite elementary divisors of $\tilde{A}(\sigma)$ at zero and are called the infinite elementary divisors of $A(\sigma)$. We denote by μ the sum of the degrees of the infinite elementary divisors i.e.

$$\mu := \sum_{j=1}^{r} \mu_j .$$

The connection between the Smith form at infinity of A(σ) and the Smith form at zero of the dual matrix is

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \sigma^q S^\infty_{A(\sigma)}(\frac{1}{\sigma}) = blockdiag\left[1, \sigma^{q-q_2}, \ldots, \sigma^{q-q_k}, \sigma^{q+\tilde{q}_{k+1}}, \ldots, \sigma^{q+\tilde{q}_r}\right]$$

So the orders of the infinite elementary divisors are given by

$$\mu_1 = 0$$

$$\mu_j = q - q_j \qquad j = 2,3,\ldots,k$$

$$\mu_j = q + \tilde{q}_j \qquad j = k+1,\ldots,r$$

Lemma 16 [Gohberg et al 1982]: Let $A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \mathbb{R}^{r\times r}[\sigma]$. Let also n,μ the sum of degrees of the finite and infinite elementary divisors of A(σ). Then

$$n + \mu = r \times q$$

where q is the highest degree among all the polynomial entries of A(σ). □

Jordan Pairs

Let $\left(C_{s_i} \in \mathbb{R}^{r\times n_i}, J_{s_i} \in \mathbb{R}^{n_i\times n_i}\right)$ be a matrix pair, where J_{s_i} is in Jordan form, corresponding to the zero s_i of $A(\sigma)$ of multiplicity n_i. This means that J_{s_i} consists of Jordan blocks with sizes equal to the partial multiplicities of s_i. This is called an *eigenpair* of $A(\sigma)$ (or a *Jordan pair*) corresponding to s_0 iff

- $rank\begin{pmatrix} C_{s_i} \\ C_{s_i} J_{s_i} \\ \ldots \\ C_{s_i} J_{s_i}^{n_i-1} \end{pmatrix} = n_i$ or equivalently written as

$$rankcol\left(C_{s_i}J_{s_i}{}^{k}\right)_{k=0}^{n_i-1}=n_i$$

- $A_qC_{s_i}J_{s_i}{}^{q}+A_{q-1}C_{s_i}J_{s_i}{}^{q-1}+...+A_1C_{s_i}J_{s_i}+A_0C_{s_i}=0$ or equivalently

$$\sum_{k=0}^{q}A_kC_{s_i}J_{s_i}{}^{k}=0$$

Taking an eigenpair for each finite eigenvalue s_i of $\mathrm{A}(\sigma)$ we can create the *finite spectral pair* of $\mathrm{A}(\sigma)$ $\left(C_F\in\mathbb{R}^{r\times n},J_F\in\mathbb{R}^{n\times n}\right)$, where

$$C_F=\left[C_1,C_2,...,C_\kappa\right]\ ;\ J_F=blockdiag\left[J_1,J_2,...,J_\kappa\right]$$

The finite spectral pair of $\mathrm{A}(\sigma)$ satisfies the same properties as the eigenpairs of $\mathrm{A}(\sigma)$ e.g.

$$rankcol\left(C_FJ_F{}^{k}\right)_{k=0}^{n_i-1}=n\ ,\qquad \sum_{k=0}^{q}A_kC_FJ_F{}^{k}=0$$

An eigenpair of the dual matrix $\tilde{\mathrm{A}}(\sigma)$ corresponding to the eigenvalue $\tilde{s}=0$ is called an *infinite spectral pair* of $\mathrm{A}(\sigma)$ and satisfies the following

$$rankcol\left(C_\infty J_\infty{}^{k}\right)_{k=0}^{\mu-1}=\mu\ ,\qquad \sum_{k=0}^{q}A_kC_\infty J_\infty{}^{q-k}=0\,.$$

An algorithm for the construction of a finite Jordan pair was given in [1].

Theorem 17 [Vardulakis 1991]: Given the matrix $A(\sigma)\in\mathbb{R}^{r\times r}$ we can construct a finite Jordan pair by following these steps:

Step 1: Compute the unimodular matrices $U_L(\sigma)\in\mathbb{R}[\sigma]^{r\times r},U_R(\sigma)\in\mathbb{R}[\sigma]^{r\times r}$ such that

$$S_{A(\sigma)}^{\mathbb{C}}(\sigma)=U_L(\sigma)A(\sigma)U_R(\sigma)$$

Step 2: Let $u_j(\sigma)\in\mathbb{R}^{r\times 1}$ $j\in r$ be the columns of $U_R(\sigma)$ and $u_j^{(q)}(\sigma)=\left(d^q/d^q\right)u_j(\sigma)$. Compute the vectors

$$\beta_{j,q}^{i}=\frac{1}{q!}u_j^{(q)}(\sigma_i),i=1,2,...,\kappa$$

where j=z,z+1,…,r and $q=0,1,...,n_{ij}-1$ and σ_i are the zeros of A(σ) with partial multiplicities

$$0\le n_{iz}\le n_{iz+1}\le...\le n_{ir}\,.$$

Step 3: Define the matrices

$$C_{i,j} = \begin{bmatrix} \beta^i_{j,0} & \beta^i_{j,1} & \cdots & \beta^i_{j,n_{i,j-2}} & \beta^i_{j,n_{i,j-1}} \end{bmatrix} \in \mathbb{R}^{r \times n_{i,j}}$$

$$J_{i,j} := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_{i,j} \times n_{i,j}}$$

and

$$C_i := \begin{bmatrix} C_{i,z} & C_{i,z+1} & \cdots & C_{i,r} \end{bmatrix} \in \mathbb{R}^{r \times m_i}$$

$$J_i := blockdiag\begin{bmatrix} J_{i,z} & J_{i,z+1} & \cdots & J_{i,r} \end{bmatrix} \in \mathbb{R}^{m_i \times m_i}$$

where $m_i = n_{i,z} + n_{i,z+1} + \cdots + n_{i,r}$.

Step 4: The pair (C,J) where

$$C := \begin{bmatrix} C_1 & C_2 & \cdots & C_\kappa \end{bmatrix} \in \mathbb{R}^{r \times n}$$

$$J := blockdiag\begin{bmatrix} J_1 & J_2 & \cdots & J_\kappa \end{bmatrix} \in \mathbb{R}^{n \times n}$$

and $n = m_1 + m_2 + \cdots + m_k = \deg\left(\prod_{j=z}^{r} f_j(\sigma) \right)$ is a Jordan pair of the polynomial matrix A(σ).

Example 18: Consider the matrix

$$\mathrm{A}(\sigma) = \begin{pmatrix} \mathrm{s+1} & \mathrm{s}^3 \\ 0 & \mathrm{s+1} \end{pmatrix}$$

We can find matrices $U_L(\sigma) \in \mathbb{R}[\sigma]^{2\times 2}, U_R(\sigma) \in \mathbb{R}[\sigma]^{2\times 2}$ such that

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = U_L(\sigma) A(\sigma) U_R(\sigma)$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & (s+1)^2 \end{pmatrix}}_{S^{\mathbb{C}}_{A(\sigma)}(\sigma)} = \begin{pmatrix} 1 & 0 \\ s+1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \mathrm{s+1} & \mathrm{s}^3 \\ 0 & \mathrm{s+1} \end{pmatrix}}_{\mathrm{A}(\sigma)} \begin{pmatrix} 3s^2 - s + 1 & -s^3 \\ -1 & s+1 \end{pmatrix}$$

so the matrix has one finite elementary divisor with multiplicity 2.

Let $u_1^{(0)} = \begin{pmatrix} -s^3 \\ s+1 \end{pmatrix}$ be the second column of $U_R(\sigma)$ that corresponds to the zero at s-1. Compute the vectors

$\beta_{1,0}^1 = \frac{1}{0!} u_1^{(0)}(-1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$\beta_{1,1}^1 = \frac{1}{1!} u_1^{(1)}(-1) = \begin{pmatrix} -3s^2 \\ 1 \end{pmatrix}_{s=-1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

So the matrix pair (C,J) with $C = C_1 = \begin{pmatrix} \beta_{1,0}^1 & \beta_{1,1}^1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ and $J = J_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is a finite Jordan pair of A(s). We can easily check that it satisfies:

$$rank\begin{pmatrix} C \\ CJ^{2-1} \end{pmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 1 \\ -1 & 4 \\ 0 & -1 \end{vmatrix}$$

$$A_3CJ^3 + A_2CJ^2 + A_1CJ + A_0C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

CHAPTER 2

SOLUTIONS OF DISCRETE TIME ALGEBRAIC DIFFERENCE EQUATIONS

In this chapter we will study the connection between the finite and infinite elementary divisors and the behavior of a system of difference equations. More specifically, we will see how f.e.ds are connected with the forward behavior of a system and how i.e.ds are connected with the backward behavior. These results have been previously studied and presented by Gohberg in 1982 and Karampetakis in 2004.

Finite elementary divisors and solutions of discrete time AR-representations

Let us assume that $\mathrm{A}(\sigma)$ has κ distinct zeros $\lambda_1, \lambda_2, ..., \lambda_\kappa$ where for simplicity of notation we assume that $\lambda_\imath \in \mathbb{C}$, $i = 1, 2, ..., \kappa$ and let

$$S_{A(\sigma)}^{C}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = blockdiag\left[1,1,...,1,\ f_z(\sigma),\ f_{z+1}(\sigma),...,\ f_r(\sigma)\right].$$

Assume that the partial multiplicities of the zeros $\lambda_\imath \in \mathbb{C}$ are $0 \le n_{i,z} \le n_{i,z+1} \le ... \le n_{i,r}$ i.e. $f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}}\ \hat{f}_j(\sigma)$ j=z,z+1,…,r with $\hat{f}_j(\lambda_i) \ne 0$. Let $u_j(\sigma) \in \mathbb{R}[\sigma]^{r\times 1}$, $j \in \mathbb{R}$ be the columns of $U_R(\sigma)$ and $u_j^{(q)}(\sigma) := (\partial^q / \partial\sigma^q)u_j(\sigma)$, $q = 0,1,...,n_{i,j-1}$. Let also

$$x_{j,q}^{i} := \frac{1}{q!}u_j^{(q)}(\lambda_i) \quad i \in 1,2,...,\kappa \text{ and j=z,z+1,…,r.}$$

Define the vector valued functions

$$\xi_{j,q}^{i}(k) := \lambda_i^k x_{j,q}^{i} + k\lambda_i^{k-1}x_{j,q-1}^{i} + ... + \binom{k}{q}\lambda_i^{k-q}x_{j,0}^{i} \quad \text{if } \lambda_i \ne 0$$

$$\xi_{j,q}^{i}(k) := \delta(k)x_{j,q}^{i} + \delta(k-1)x_{j,q-1}^{i} + ... + \delta(k-q)x_{j,0}^{i} \quad \text{if } \lambda_i = 0$$

$$i \in k\ ;\ \text{j=z,z+1,…,r;}\quad q = 0,1,...,n_{i,j-1}$$

where by δ(k) we denote the known Kronecker delta function. Let

$$\Psi_{i,j}(k) := \left[\xi_{j,0}^{i}(k) \quad \xi_{j,1}^{i}(k) \quad ... \quad \xi_{j,n_{i,j-2}}^{i}(k) \quad \xi_{j,n_{i,j-1}}^{i}(k)\right]$$

$$C_{i,j} := \left[x_{j,0}^{i} \quad x_{j,1}^{i} \quad ... \quad x_{j,n_{i,j-2}}^{i} \quad x_{j,n_{i,j-1}}^{i}\right]$$

$$J_{i,j} := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_{i,j} \times n_{i,j}}$$

where $i = 1, 2, ..., \kappa$, j=z,z+1,…,r and

$$\Psi_i^F(k) := \left[\Psi_{i,z}(k) \quad \Psi_{i,z+1}(k) \quad \cdots \quad \Psi_{i,r-1}(k) \quad \Psi_{i,r}(k) \right]$$

$$C_i^F := \left[C_{i,z}(k) \quad C_{i,z+1}(k) \quad \cdots \quad C_{i,r-1}(k) \quad C_{i,r}(k) \right]$$

$$J_i^F = blockdiag\left[J_{i,z}(k) \quad J_{i,z+1}(k) \quad \cdots \quad J_{i,r-1}(k) \quad J_{i,r}(k) \right]$$

Finally let

$$\Psi_F^D(k) := \left[\Psi_1^F(k) \quad \Psi_2^F(k) \quad \cdots \quad \Psi_{\kappa-1}^F(k) \quad \Psi_\kappa^F(k) \right]$$

$$C_F^D := \left[C_1^F(k) \quad C_2^F(k) \quad \cdots \quad C_{\kappa-1}^F(k) \quad C_\kappa^F(k) \right]$$

$$J_F^D = blockdiag\left[J_1^F(k) \quad J_2^F(k) \quad \cdots \quad J_{\kappa-1}^F(k) \quad J_\kappa^F(k) \right]$$

The solution space of the system (1) is:

$$B_F^D = \left\langle \Psi_F^D(k) \right\rangle = \left\langle C_F^D \left(J_F^D \right)^k \right\rangle$$

Example 19: Consider the matrix $P(\sigma) = \begin{pmatrix} \sigma^2 & \sigma - 1 \\ \sigma & \sigma - 1 \end{pmatrix}$. It's Smith form is

$$S_{A(\sigma)}^{\mathbb{C}}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma(\sigma-1)^2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & -\sigma^2 + \sigma - 1 \end{pmatrix}}_{U_L(\sigma)} \begin{pmatrix} \sigma^2 & \sigma - 1 \\ \sigma & \sigma - 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & \sigma - 1 \\ 1 & -\sigma \end{pmatrix}}_{U_R(\sigma)}$$

The second column $u_2(\sigma) = \begin{pmatrix} \sigma - 1 \\ -\sigma \end{pmatrix}$ corresponds to the finite elementary divisors σ and $(\sigma-1)^2$.

Thus $\lambda_1 = 0$ with $\sigma_{1,2} = 1$ and $\lambda_2 = 1$ with $\sigma_{2,2} = 2$.

We have:

$$\beta_{2,0}^1 = \frac{1}{0!} u_2^{(0)}(\lambda_1) = \begin{pmatrix} 0-1 \\ -0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

for the first zero of A(σ) and for the second zero of A(σ):

$$\beta_{2,0}^{2}=\frac{1}{0!}u_2^{(0)}(\lambda_2)=\begin{pmatrix}1-1\\-1\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$\beta_{2,1}^{2}=\frac{1}{1!}u_2^{(1)}(\lambda_2)=\begin{pmatrix}\sigma-1\\-\sigma\end{pmatrix}_{\sigma=1}^{(1)}=\begin{pmatrix}1\\-1\end{pmatrix}$$

Create the pairs

$$(C_1,J_1)=\begin{pmatrix}\beta_{2,0}^{1} & 0\end{pmatrix}=\left(\begin{pmatrix}-1\\0\end{pmatrix}\quad(0)\right) \text{ and } (C_2,J_2)=\left(\begin{pmatrix}\beta_{2,0}^{2} & \beta_{2,1}^{2}\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)=\left(\begin{pmatrix}0&1\\-1&-1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)$$

Now we create the matrices

$$C_F^D=\begin{pmatrix}C_1 & C_2\end{pmatrix}=\begin{pmatrix}-1&0&1\\0&-1&-1\end{pmatrix} \text{ and } J_F^D=blockdiag\begin{pmatrix}J_1 & J_2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

We also define the functions

$$\tilde{\beta}_{2,0}^{1}=\begin{pmatrix}-1\\0\end{pmatrix}\delta(k)\rightarrow\Psi_1(k)=\left(\begin{pmatrix}-1\\0\end{pmatrix}\delta(k)\right)$$

$$\left.\begin{array}{l}\tilde{\beta}_{2,0}^{2}=\begin{pmatrix}1\\-1\end{pmatrix}1^{k}=\begin{pmatrix}1\\-1\end{pmatrix}\\ \tilde{\beta}_{2,1}^{2}=\begin{pmatrix}0\\-1\end{pmatrix}+k\begin{pmatrix}1\\-1\end{pmatrix}1^{k-1}=\begin{pmatrix}0\\-1\end{pmatrix}+k\begin{pmatrix}1\\-1\end{pmatrix}\end{array}\right\rangle\Psi_2(k)=\left(\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}+k\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$\Psi_2(k)=\left(\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}k\\-1-k\end{pmatrix}\right)$$

So the solution space of A(σ)β(k)=0 is spanned by the vectors:

$$\Psi_F^D(k)=\begin{pmatrix}\Psi_1(k) & \Psi_2(k)\end{pmatrix}=\left(\begin{pmatrix}-1\\0\end{pmatrix}\delta(k),\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}k\\-1-k\end{pmatrix}\right)=C_F^D\left(J_F^D\right)^{k}$$

Infinite elementary divisors and solutions of discrete time AR-representations

Let $\tilde{U}_L(\sigma)\in\mathbb{R}^{r\times r},\tilde{U}_R(\sigma)\in\mathbb{R}^{r\times r}$ be rational matrices having no poles and zeros at σ=0 such that

$$S_{\tilde{A}(\sigma)}^{0}(\sigma)=\tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma)=blockdiag\left[\sigma^{\mu_1},\sigma^{\mu_2},\ldots,\sigma^{\mu_r}\right]=$$

$$= blockdiag\left[1, \sigma^{q-q_2}, \ldots, \sigma^{q-q_k}, \sigma^{q+\tilde{q}_{k+1}}, \ldots, \sigma^{q+\tilde{q}_r}\right]$$

Where $S^0_{\tilde{A}(\sigma)}(\sigma)$ is the Smith form of $\tilde{A}(\sigma)$ at zero. Let also $\tilde{U}_R(\sigma) = \left[\tilde{u}_1(\sigma) \quad \tilde{u}_2(\sigma) \quad \cdots \quad \tilde{u}_r(\sigma)\right]$ where $\tilde{u}_j(\sigma) \in R(\sigma)^{r\times 1}$ and $\tilde{u}^{(i)}_j(\sigma)$, $\tilde{A}^{(i)}(\sigma)$ be the i-th derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ respectively, for i=0,1,…, μ_j and $j \in r$. Define

$x_{j,i} := \frac{1}{i!}\tilde{u}^{(i)}_j(0)$ for $i = 0,1,..., \mu_j$ and $j \in r$. Then for initial conditions

$$\begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} = \begin{bmatrix} x_{j,i} \\ \vdots \\ \vdots \\ x_{q_j-1} \end{bmatrix}$$

we obtain respectively the linearly independent backward solutions

$$\xi^B_{j,i}(k) := x_{j,i}\delta(N-k) + x_{j,i-1}\delta(N-(k+1)) + ... + x_{j,0}\delta(N-(k+i)) \quad i=0,1,\ldots,\mu_j;\ j \in r$$

Let

$$\Psi^B_j(k) := \left[\xi^B_{j,0}(k) \quad \xi^B_{j,1}(k) \quad \cdots \quad \xi^B_{j,\mu_{j-2}}(k) \quad \xi^B_{j,\mu_{j-1}}(k)\right]$$

$$C^B_j = \left[x_{j,0} \quad x_{j,1} \quad \cdots \quad x_{j,\mu_j-1}\right]$$

$$J^B_j := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\mu_j \times \mu_j} \quad \text{where } j \in r \text{ and}$$

$$\Psi^D_B(k) := \left[\Psi^D_k(k) \quad \Psi^D_{k+1}(k) \quad \cdots \quad \Psi^D_r(k)\right]$$

$$C^D_B(k) := \left[C^D_k(k) \quad C^D_{k+1}(k) \quad \cdots \quad C^D_r(k)\right]$$

$$J^D_B(k) := blockdiag\left[J^D_k(k) \quad J^D_{k+1}(k) \quad \cdots \quad J^D_r(k)\right]$$

where

$$\mu := \sum_{j=1}^{r} \mu_j$$

The solution space spanned by the i.e.d. of (1) is

$$B_B^D = \left\langle \Psi_B^D(k) \right\rangle = \left\langle C_B^D \left(J_B^D \right)^{N-k} \right\rangle$$

Example 20: Consider the polynomial matrix

$$A(\sigma) = \begin{pmatrix} \sigma+1 & \sigma \\ \sigma^2+1 & \sigma^2+\sigma \end{pmatrix}$$

Its dual matrix is

$$\tilde{A}(\sigma) = \sigma^2 A\left(\frac{1}{\sigma}\right) = \begin{pmatrix} \sigma+\sigma^2 & \sigma \\ 1+\sigma^2 & 1+\sigma \end{pmatrix}$$

and has the Smith form at zero

$$S_{\tilde{A}(\sigma)}^0 = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1+\sigma & -\sigma \end{pmatrix}}_{\tilde{U}_L} \begin{pmatrix} \sigma+\sigma^2 & \sigma \\ 1+\sigma^2 & 1+\sigma \end{pmatrix} \underbrace{\begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2}+\frac{1}{2}\sigma \end{pmatrix}}_{\tilde{U}_R}$$

Therefore $\tilde{A}(\sigma)$ has a zero at σ=0 of multiplicity 2. Let $\tilde{u}_2(\sigma) = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2}\sigma \end{pmatrix}$ be the second column of $\tilde{U}_R$. Define

$$x_{2,0} = \frac{1}{0!}\tilde{u}_2(0) = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$x_{2,1} = \frac{1}{1!}\tilde{u}_2^{(1)}(0) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

and

$$C_B^D = \begin{pmatrix} x_{2,0} & x_{2,1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$J_B^D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The solution space B_B^D is spanned by the columns of the matrix:

$$\Psi_B^D(k) = \left\langle C_B^D \left(J_B^D\right)^{N-k} \right\rangle = \left\langle \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \delta(N-k) & \delta\left(N-(k+1)\right) \\ 0 & \delta(N-k) \end{pmatrix} \right\rangle =$$

$$= \left\langle \begin{pmatrix} \frac{1}{2}\delta(N-k) \\ -\frac{1}{2}\delta(N-k) \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2}\delta\left(N-(k+1)\right) \\ -\frac{1}{2}\delta\left(N-(k+1)\right) + \frac{1}{2}\delta(N-k) \end{pmatrix} \right\rangle$$

□

CHAPTER 3

CONSTRUCTION OF A SYSTEM OF ALGEBRAIC DIFFERENCE EQUATIONS WITH GIVEN FORWARD/BACKWARD BEHAVIOR

In this section we are going to study the inverse problem i.e. given a specific forward and backward behavior, how to construct the polynomial matrix A(σ), so that the AR-representation A(σ)β(k)=0 has the given behavior. On this subject we already have results regarding the forward behavior but there are no results concerning the backward one. It must be mentioned though that the analog problem for continuous time systems has been examined in [2]. We will first present and produce results concerning the forward and backward behavior separately and then we will study the case where both a forward and backward behavior is given.

Theorem 21 [Gohberg 1982, Karampetakis 2004]: Suppose that a finite number of functions of the form

$$\beta_i(k) := \lambda_i^k \beta_{q_i,i} + k\lambda_i^{k-1}\beta_{q_i-1,i} + \ldots + \begin{pmatrix} k \\ k-q_i \end{pmatrix} \lambda_i^{k-q_i}\beta_{0,i} = \sum_{j=0}^{q_i} \begin{pmatrix} k \\ k-q_i+j \end{pmatrix} \lambda_i^{k-q_i+j}\beta_{j,i}, \quad \lambda_i \neq 0$$

$$\left(\beta_{j,q_j}(k) = \delta(k)x_{j,q_j-1} + \delta(k-1)x_{j,q_j-2} + \ldots + \delta\left(k-(q_j-1)\right)x_{j,0}, \left[\lambda_i = 0\right]\right)$$

are given, where $\beta_{j,i} \in \mathbb{C}$, $0 \le j \le q_i$ and $i = 1,2,\ldots,l$. Let

$$C_i := \begin{bmatrix} \beta_{0,i} & \beta_{1,i} & \cdots & \beta_{q_i-1,i} & \beta_{q_i,i} \end{bmatrix} \in \mathbb{R}^{1\times(q_i+1)},\ J_i := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{(q_i+1)\times(q_i+1)}$$

and

$$C := \begin{bmatrix} C_1 & C_2 & \cdots & C_{l-1} & C_l \end{bmatrix} \in \mathbb{R}^{1\times n},\ J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_l \end{bmatrix} \in \mathbb{R}^{n\times n}$$

where $n := \sum_{i=1}^{l}(q_i+1)$

Let a be a complex number different than λ_i and define

$$A(\sigma) = I_r - C(J - aI_n)^{-q}\left\{(\sigma - a)V_q + (\sigma - a)^2 V_{q-1} + ... + (\sigma - a)^q V_1\right\}$$

where $q = ind(C,J)$ and $\begin{bmatrix} V_1 & V_2 & \cdots & V_q \end{bmatrix}$ is the generalised inverse of $S_{1-q} = \begin{bmatrix} C \\ C(J-aI_n)^{-1} \\ \vdots \\ C(J-aI_n)^{1-q} \end{bmatrix}$.

Then $\beta_i(k)$ are solutions of $A(\sigma)\beta(k) = 0$. Furthermore, q is the minimal possible lag of any $n \times n$ matrix with this property.

What we need to mention here is that although we created an AR-representation that has the solution $\beta_i(k)$, the vectors $\beta_i(k)$ does not necessarily span the whole solution space of A(σ). This depends on the dimensions of the matrix pair (C,J). According to Lemma 16, the sum of the i.e.ds and the f.e.ds is $n + \mu = r \cdot q$. This means that in order for a pair (C,J) to fully describe a system, in terms of its finite and infinite elementary divisors, it must be of dimensions $n \times rq$ and $rq \times rq$. So in case that the matrix pair has these dimensions, then the vectors $\beta_i(k)$ span the hole solution space of A(σ). But in any other case, the system will exhibit some extra linearly independent behavior. This holds true for all the algorithms for construction of a system with given behavior that will be presented in this paper.

For an analytic proof of Theorem 21, one may refer to [Gohberg et al, 1981].

Example 22: We are looking for the matrix A(σ), knowing that the solution space is

$$B^{\mathbb{C}} = \left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} 2^k \quad \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} 2^k + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} k2^k \quad \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} 2^k + \begin{pmatrix} 2 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} k2^k + \begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \left(k^2 - k\right) 2^k \right\rangle$$

. Doing some calculations we can rewrite it as

$$B^{\mathbb{C}} = \left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} 2^k \quad \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} 2^k + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} k2^{k-1} \quad \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} 2^k + \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} k2^{k-1} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \frac{k(k-1)}{2} 2^{k-2} \right\rangle$$

So now it's easier to see that

$$\beta_1(k)=\underbrace{\begin{pmatrix}2\\3\\2\end{pmatrix}}_{\beta_{1,2}}2^k+\underbrace{\begin{pmatrix}4\\1\\1\end{pmatrix}}_{\beta_{1,1}}k2^{k-1}+\underbrace{\begin{pmatrix}2\\0\\0\end{pmatrix}}_{\beta_{1,0}}k(k-1)2^{k-2}$$

Define

$$C=C_1=\begin{pmatrix}\beta_{1,0} & \beta_{1,1} & \beta_{1,2}\end{pmatrix}=\begin{pmatrix}2&4&2\\0&1&3\\0&1&2\end{pmatrix},\ J=\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}$$

We now must find q=ind(C,J). We can see that $\det(\mathrm{C})=-2\neq 0$, so q=1.

$$V_1=C^{-1}=\begin{pmatrix}\frac{1}{2}&3&-5\\0&-2&3\\0&1&-1\end{pmatrix}$$

For $a=1\neq 2$ we have

$$A(\sigma)=\mathrm{I}_3-C\left(J-1I_3\right)^{-1}(\sigma-1)V_1=$$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}2&4&2\\0&1&3\\0&1&2\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}^{-1}(\sigma-1)\begin{pmatrix}\frac{1}{2}&3&-5\\0&-2&3\\0&1&-1\end{pmatrix}=\begin{pmatrix}2-\sigma&-2\sigma+2&4\sigma-4\\0&1&1-\sigma\\0&\sigma-1&-2\sigma+3\end{pmatrix}$$

with $S_{A(\sigma)}^{\mathbb{C}}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&(\sigma-2)^3\end{pmatrix}$. The pair (C,J) is a finite Jordan pair for the matrix A(σ) that we created, because indeed it satisfies the 2 properties of a Jordan Pair, which are

$$A_1CJ+A_0C=\begin{pmatrix}-1&-2&4\\0&0&-1\\0&1&-2\end{pmatrix}\begin{pmatrix}2&4&2\\0&1&3\\0&1&2\end{pmatrix}\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}+\begin{pmatrix}2&2&-4\\0&1&1\\0&-1&3\end{pmatrix}\begin{pmatrix}2&4&2\\0&1&3\\0&1&2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Of course, the first property $rank\begin{pmatrix} C_{s_i} \\ C_{s_i}J_{s_i} \\ \ldots \\ C_{s_i}J_{s_i}^{n_i-1} \end{pmatrix} = n_i$, (in this example $rank\begin{pmatrix} C \\ CJ \\ CJ^2 \end{pmatrix} = 3$) is always satisfied, because it is a necessary condition in the creation of the matrix A(σ).

□

Example 23: We are looking for the polynomial matrix A(σ), with solution space spanned by

$$\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} k \right\rangle$$

We can see that $\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\beta_{1,1}} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\beta_{1,0}} k$

Define

$$C = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \text{ and } J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Now we must find q=ind(C,J). We can see that $\det(C) = -1 \neq 0$, so q=1

$$V_1 = C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

For $a = 2 \neq 1$ we have

$$A(\sigma) = I_2 - C\left(J - 2I_2\right)^{-1}(\sigma - 2)V_1 =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} (\sigma - 2) \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sigma - 1 & 2 - \sigma \\ 0 & \sigma - 1 \end{pmatrix}$$

with Smith form

$$S_{A(\sigma)}^{\mathbb{C}} = \begin{pmatrix} 1 & 0 \\ 0 & (\sigma - 1)^2 \end{pmatrix}.$$

Just by looking at the Smith form we can be sure that our algorithm works, but we can also check that that the finite Jordan pair (C, J) satisfies

- $A_1CJ + A_0C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- $rank\begin{pmatrix} C \\ CJ \end{pmatrix} = 2$

□

Example 24: Let

$$x(k) = \underset{x_{1,2}}{\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}}\delta(k) + \underset{x_{1,1}}{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}\delta(k-1) + \underset{x_{1,0}}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}\delta(k-2) \equiv \underset{x_{1,2}}{\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}}\delta(k) + \frac{k}{1}\underset{x_{1,1}}{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}\delta(k-1) + \frac{k(k-1)}{2}\underset{x_{1,0}}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}\delta(k-2) \quad .$$

We want to find an AR-representation that has x(k) as its solution. We begin by creating the pair

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ -1 & 3 & 2 \end{pmatrix}, J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we must find the lag of the matrix. We know that q=ind(C,J).

Since detC=0, the assumption that q=1 is rejected.

Let q=2, the matrix

$$S = \begin{pmatrix} C \\ CJ^{2-1} \end{pmatrix} = \begin{pmatrix} C \\ CJ \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ -1 & 3 & 2 \\ \hline 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & 3 \end{pmatrix}$$

has full column rank, so q=2. Let a=1 and compute A(σ) by

$$\mathrm{A}(\sigma) = \mathrm{I}_3 - C\left(J - I_3\right)^{-2}\left\{(\sigma-1)V_2 + (\sigma-1)^2 V_1\right\}$$

where $\begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} C \\ C(J-1)^{-1} \end{pmatrix}^{-1} = \left(\begin{array}{ccc|ccc} \frac{345}{2656} & \frac{465}{2656} & \frac{15}{1328} & \frac{37}{664} & -\frac{331}{2656} & \frac{897}{2656} \\ \frac{497}{2656} & -\frac{23}{2656} & \frac{599}{1328} & \frac{61}{664} & -\frac{115}{2656} & \frac{761}{2656} \\ -\frac{65}{664} & -\frac{1}{664} & -\frac{75}{332} & -\frac{19}{166} & -\frac{5}{664} & -\frac{169}{664} \end{array}\right)$

A(σ)=

$$\begin{pmatrix} \frac{1007+1382s+267s^2}{2656} & \frac{-1324+1657s-333s^2}{2656} & \frac{-1641+463s+1178s^2}{2656} \\ \frac{-2143+2458s-315s^2}{2656} & \frac{428+2999s-771s^2}{2656} & \frac{11\left(-117+67s+50s^2\right)}{2656} \\ \frac{1}{664}\left(-69-218s+287s^2\right) & \frac{1}{664}\left(156-283s+127s^2\right) & \frac{1}{664}\left(243-37s+458s^2\right) \end{pmatrix}$$

The system generated by the equation A(σ)β(k)=0 has indeed the solution we want. We can check that (C,J) is a Jordan pair of A(σ) corresponding to the f.e.d. zero, because

$$A_2CJ^2+A_1CJ+A_0C=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ;\ rank\begin{pmatrix} C \\ CJ \\ CJ^2 \end{pmatrix}=3$$

On the other hand, if we take the determinant of matrix A(σ), we see that

$$\det\left(A(\sigma)\right)=-\frac{1}{228}\sigma^3(-331+103\sigma)\,.$$

The determinant as we know has the same elements as the Smith form of A(σ). This means that our system has an extra solution corresponding to the f.e.d. $\sigma=\frac{331}{103}$

That is the reason we cannot state that $x(k)$ can span the whole solution space of A(σ).

Based on theory (Lemma 16 and [Gohberg et al 1981]), in order for the matrix pair (C,J) to contain of the spectral information for A(σ)β(k)=0, it must be of dimensions:

$$C: r\times rq \Rightarrow 3\times 6$$
$$J: rq\times rq \Rightarrow 6\times 6$$

So what one can do in order to control the behavior of the system, and not let it be arbitrarily created, is to create the pair

$$\bullet C=\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & a & b & c \\ 2 & 1 & 3 & d & e & f \\ -1 & 3 & 2 & g & h & i \end{array}\right);J=\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right]$$

and add a behavior of our choise. Of course there are other possible forms for J, like

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Another way to create a system matrix A(σ) having $x(k)$ as the only behavior, we need to choose the right value for a.

Using Mathematica, we compute the pseudoinverse matrix, but now considering ‘a’ (named ‘b’ in Mathematica) as a variable. The string of commands for these computations in Mathematica is

```
c={{3, 2, 1}, {3, 1, 2}, {2, 3, -1}}
j = {{0, 1, 0}, {0, 0, 1}, {0, 0, 0}}
MatrixRank[ArrayFlatten[{{c}, {c.j}}]]
m = ArrayFlatten[{{c}, {c.MatrixPower[j - b IdentityMatrix[3], -1]}}]
m1 = Simplify[PseudoInverse[m], b \[Element] Reals]
v1 = Take[m1, {1, 3}, {1, 3}]
v2 = Take[m1, {1, 3}, {4, 6}]
a = IdentityMatrix[3] - c.MatrixPower[j - b IdentityMatrix[3], -2].((s - b) v2 + (s - b)^2 v1) // Simplify
```

The result of this computation is too long to be presented here. We proceed to compute the determinant of A(σ).

```
Det[a] // Factor
```

which gives the result

$$\frac{s^3\left(88+111b+132b^2-37s-66bs\right)}{2b^3\left(44+37b+33b^2\right)}$$

In order for our system to have only zero as an i.e.d. the quantity $-37s-66as$ must be zero, thus

```
Solve[-37 s - 66 b s == 0, b]
```

$\{\{b \to -(\frac{37}{66})\}\}$

Starting the algorithm again and evaluating A(σ) for the new value of a (b in Mathematica) we finally get

A(σ)=

$$\begin{bmatrix} 9739771035-16744593164s-1477832928s^2 & -37259417249-66950044420s-869238084s^2 & 41279469321+72512689984s-1999546692s^2 \\ -9359753063-19882245334s-5683972800s^2 & -28649926839-86269262900s-3343223400s^2 & 57014519853+97390199054s-7690564200s^2 \\ -5219970989-5933641826s+6025011168s^2 & -8084696581-12434665480s+3543816804s^2 & 19957001355+6879278206s+8151998052s^2 \end{bmatrix}$$

with $det(A) = -10.6\cdot 10^9 s^3$, which means that the only f.e.d. is zero.

□

We shall now extend this theorem to the case of backward solutions.

As we have already shown, the smith form of the dual matrix of A(σ) at zero is

$S^0_{\tilde{A}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = blockdiag\left(1, w^{q-q_2}, \dots, w^{q-q_k}, w^{q+\tilde{q}_{k+1}}, \dots, w^{q+\tilde{q}_r}\right) \rightarrow$

$\rightarrow \tilde{A}(\sigma)\tilde{u}_j(\sigma) = v_j(\sigma) w^{\mu_j}$ j=2,…,r

$\mu_j = q - q_{j,}$, j=2,…,k ; $\mu_j = q + \tilde{q}_{j,}$, j=k+1,…,r and $\tilde{u}_j(\sigma), v_j(\sigma)$ are the j-th columns of $\tilde{U}_R(\sigma)$ and $\tilde{U}_L(\sigma)^{-1}$ respectively.

Lemma 25: Let $\tilde{u}_j^{(i)}(\sigma)$, $\tilde{A}^{(i)}(\sigma)$ be the i-th derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ with respect to σ for i=0,1,…, μ_j and j=2,…,r. The vectors $x_{j,i}$ as we have already seen, defined by:

$x_{j,i} := \frac{1}{i!}\tilde{u}_j^{(i)}(0)$ for i=0,1,…, $q+q_j-1$ and j=2,…,r.

form Jordan Chains corresponding to $\tilde{A}(\sigma)$ and thus satisfy the conditions

$$\begin{aligned} &\tilde{A}(0)x_{j0} = 0 \\ &\tilde{A}^1(0)x_{j0} + \tilde{A}(0)x_{j1} = 0 \\ &\dots \\ &\frac{1}{(q+q_j-1)!}\tilde{A}^{q+q_j-1}(0)x_{j0} + \dots + \tilde{A}(0)x_{jq+q_j-1} = 0 \end{aligned} \qquad \textbf{(4)}$$

Proof: Since $\tilde{U}_R(\sigma)$ has no poles or zeros at w=0, $\tilde{u}_j(0) \neq 0$ and for w=0

$$\tilde{A}(0)\tilde{u}_j(0) = 0, \qquad j=2,\dots,r \qquad \textbf{(5)}$$

taking the first derivative of (5) with respect to w we have

$\tilde{A}^{(1)}(w)\tilde{u}_j(w)+\tilde{A}(w)\tilde{u}_j^{(1)}(w)=v_j^{(1)}(w)w^{\mu_j}+v_j(w)\mu_j w^{\mu_j-1}$ which for w=0 gives

$$\tilde{A}^{(1)}(0)\tilde{u}_j(0)+\tilde{A}(0)\tilde{u}_j^{(1)}(0)=0, \qquad j=2\ldots,r$$

In the same fashion, we take the derivative of the above equation and evaluate the result for w=0. We obtain:

$$\tilde{A}^{(2)}(0)\tilde{u}_j(0)+2\tilde{A}^{(1)}(0)\tilde{u}_j^{(1)}(0)+\tilde{A}(0)\tilde{u}_j^{(2)}(0)=0\rightarrow$$

$$\tilde{A}^{(2)}(0)x_{j,0}+2\tilde{A}^{(1)}(0)x_{j,1}+\tilde{A}(0)x_{j,2}=0 \qquad j=2,\ldots,r$$

Continuing this procedure until the $(q+q_j-1)$ derivative of (5) we obtain the equations of our lemma. □

Now from (3) we can easily get that

$$\tilde{A}^{(p)}(0)=p!A_{q-p} \qquad p=1,2,,\ldots,q$$

$$\tilde{A}^{(p)}(0)=0 \qquad p=q+1,...,q+q_j-1$$

So the equations (6) of lemma transform to

$$\begin{aligned}
&\tilde{A}(0)x_{j,0}=0\\
&\tilde{A}^{(1)}(0)x_{j,0}+\tilde{A}(0)x_{j,1}=0\\
&\vdots\\
&\frac{1}{q!}\tilde{A}^{(q)}(0)x_{j,0}+\frac{1}{(q-1)!}\tilde{A}^{(q-1)}(0)x_{j,1}+...+\tilde{A}(0)x_{j,q}=0\\
&\frac{1}{q!}\tilde{A}^{(q)}(0)x_{j,1}+\frac{1}{(q-1)!}\tilde{A}^{(q-1)}(0)x_{j,2}+...+\tilde{A}^{(1)}(0)x_{j,q}+\tilde{A}(0)x_{j,q+1}=0\\
&\vdots\\
&\frac{1}{q!}\tilde{A}^{(q)}(0)x_{j,q_j-1}+\frac{1}{(q-1)!}\tilde{A}^{(q-1)}(0)x_{j,q_j}+...+\tilde{A}^{(1)}(0)x_{j,q+q_j-2}+\tilde{A}(0)x_{j,q+q_j-1}=0
\end{aligned}$$

for j=2,…,r

These equations can be summarized in matrix form as

$$\begin{pmatrix} A_q & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ A_{q-1} & A_q & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_q & 0 & 0 & 0 & 0 \\ A_0 & A_1 & A_2 & \cdots & A_{q-1} & A_q & 0 & 0 & 0 \\ 0 & A_0 & A_1 & \cdots & A_{q-2} & A_{q-1} & A_q & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_0 & A_1 & A_2 & \cdots & A_{q-1} & A_q \end{pmatrix} \begin{pmatrix} x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,q-1} \\ x_{j,q} \\ x_{j,q+1} \\ \vdots \\ x_{j,q+q_j-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \textbf{(6)}$$

□

Theorem 26: The vector $\beta_j^{\infty}(k) = x_{j,q+q_j-1}\delta(N-k) + x_{j,q+q_j-2}\delta(N-k-1) + ... + x_{j,0}\delta(N-k-q-q_j+1)$ **(7)**

is a solution of the AR-representation A(σ)β(k)=0 iff

$$\tilde{\beta}_j(k) = x_{j,0}\delta(k-q-q_j+1) + x_{j,1}\delta(k-q-q_j+2) + \cdots + x_{j,q_j-1}\delta(k-q) + \cdots + x_{j,q+q_j-1}\delta(k) \quad \textbf{(8)}$$

is a solution of the dual homogenous system $\tilde{A}(\sigma)\tilde{\beta}(k) = 0$

Proof: ($\Rightarrow$) First assume that (7) is a solution of (1), then the equations (6) hold true. We will show that (8) is a solution of (3), i.e.

$$\tilde{A}(\sigma)\tilde{\beta}(k) = 0$$

. or equivalently, taking the Z-transform, that

$$\tilde{A}(z)\tilde{\beta}(z) = \hat{b}(z)$$

where $\hat{b}(z)$ is defined by the initial values of $\tilde{\beta}(k)$,

$$\hat{b}(z) = \begin{pmatrix} z^q I_r & z^{q-1} I_r & \cdots & z I_r \end{pmatrix} \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} \tilde{\beta}(0) \\ \tilde{\beta}(1) \\ \vdots \\ \tilde{\beta}(q-1) \end{pmatrix}$$

$$\Rightarrow \tilde{\mathrm{A}}(z)\tilde{\beta}(z)=\left(z^{q}A_0+z^{q-1}A_1+\cdots zA_{q-1}+A_q\right)\left(x_{j,0}z^{-q-q_j+1}+x_{j,1}z^{-q-q_j+2}+\cdots+x_{j,q_j-1}z^{-q}+\cdots+x_{j,q+q_j-1}\right)=$$

$$=\begin{pmatrix} z^{q}I_r & z^{q-1}I_r & \cdots & zI_r & I_r & z^{-1}I_r & \cdots & z^{-q-q_j+1}I_r \end{pmatrix}\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_q & A_{q-1} & \ddots & A_0 \\ 0 & A_q & \ddots & A_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ \vdots \\ x_{j,q_j-1} \\ \vdots \\ x_{j,1} \\ x_{j,0} \end{pmatrix}=$$

$$=\begin{pmatrix} z^{q}I_r & z^{q-1}I_r & \cdots & zI_r \end{pmatrix}\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ x_{j,q+q_j-2} \\ \vdots \\ x_{j,q_j} \end{pmatrix}+$$

$$+\begin{pmatrix} I_r & z^{-1}I_r & \cdots & z^{-q}I_r & \cdots & z^{-q-q_j-1}I_r \end{pmatrix}\begin{pmatrix} A_q & A_{q-1} & \cdots & A_0 & 0 & \cdots & 0 \\ 0 & A_q & \cdots & A_1 & A_0 & \ddots & \vdots \\ 0 & 0 & \ddots & \vdots & \vdots & \ddots & 0 \\ \vdots & \ddots & \ddots & A_q & A_{q-1} & \cdots & A_0 \\ \vdots & \vdots & \ddots & 0 & A_q & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & A_q \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ \vdots \\ \vdots \\ x_{j,q} \\ \vdots \\ x_{j,1} \\ x_{j,0} \end{pmatrix}=$$

$$=\begin{pmatrix} z^{q}I_r & z^{q-1}I_r & \cdots & zI_r \end{pmatrix}\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ x_{j,q+q_j-2} \\ \vdots \\ x_{j,q_j} \end{pmatrix}+$$

$$+\left(z^{-q-q_j+1}I_r \quad z^{-q-q_j+2}I_r \quad \cdots \quad z^{-q}I_r \quad \cdots \quad I_r\right)\underbrace{\begin{pmatrix} A_q & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ A_{q-1} & A_q & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_q & 0 & 0 & 0 & 0 \\ A_0 & A_1 & A_2 & \cdots & A_{q-1} & A_q & 0 & 0 & 0 \\ 0 & A_0 & A_1 & \cdots & A_{q-2} & A_{q-1} & A_q & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_0 & A_1 & A_2 & \cdots & A_{q-1} & A_q \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ \vdots \\ \vdots \\ x_{j,q} \\ \vdots \\ x_{j,1} \\ x_{j,0} \end{pmatrix}}_{=0 \quad (6)}$$

$$\Rightarrow \left(z^{q}I_r \quad z^{q-1}I_r \quad \cdots \quad zI_r\right)\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ x_{j,q+q_j-2} \\ \vdots \\ x_{j,q_j} \end{pmatrix} = \hat{b}(z)$$

So we have that

$$\left(z^{q}I_r \quad z^{q-1}I_r \quad \cdots \quad zI_r\right)\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q+q_j-1} \\ x_{j,q+q_j-2} \\ \vdots \\ x_{j,q_j} \end{pmatrix} = \left(z^{q}I_r \quad z^{q-1}I_r \quad \cdots \quad zI_r\right)\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{pmatrix}\begin{pmatrix} \tilde{\beta}(0) \\ \tilde{\beta}(1) \\ \vdots \\ \tilde{\beta}(q-1) \end{pmatrix}$$

This equation holds true, because from the way we have defined $\tilde{\beta}_j(k)$, we can see that

$$\tilde{\beta}_j(0) = x_{j,0}0 + x_{j,1}0 + \ldots + x_{j,q+q_j-1}\delta(0) = x_{j,q+q_j-1}$$

$$\tilde{\beta}_j(1) = 0 + \ldots + x_{j,q+q_j-2}\delta(1-1) + \ldots + 0 = x_{j,q+q_j-2}$$

$$\cdots$$

$$\tilde{\beta}_j(q-1) = 0 + \ldots + x_{j,q_j}\delta(q-1-q+1) + \ldots + 0 = x_{j,q_j}$$

Thus, the right and left side of the equation are equal, so $\tilde{\beta}_j(k)$ is a solution of $\tilde{A}(\sigma)\tilde{\beta}(k) = 0$.

($\Leftarrow$) Now we will show that if $\tilde{\beta}_j(k)$ is a solution of $\tilde{A}(\sigma)\tilde{\beta}(k) = 0$, then $\beta_j^{\infty}(k)$ is a solution of A(σ)β(k)=0.

We will apply the Z-transform $\beta(z) = \sum_{-\infty}^{N} \beta(k)z^{-k}$.

$Z\left\{\beta(k+1)\right\} = \sum_{-\infty}^{N} \beta(k+1)z^{-k}$. Let k+1=m and we have

$$\sum_{-\infty}^{N}\beta(k+1)z^{-k}=\sum_{-\infty}^{N+1}\beta(m)z^{-m+1}=$$

$$=z\sum_{-\infty}^{N}\beta(m)z^{-m}+\beta(N+1)z^{-N}=$$

$$=zB(z)+\beta(N+1)z^{-N}$$

Likewise, we have

$$Z\left\{\beta(k+2)\right\}=z^{2}B(z)+\beta(N+1)z^{-N+1}+\beta(N+2)z^{-N}$$

....

$$Z\left\{\beta(k+q)\right\}=z^{q}B(z)+\beta(N+1)z^{-N+q-1}+...+\beta(N+q)z^{-N}$$

So applying this transform to the equation $A(\sigma)\beta_j^{\infty}(k)=0$ we have

$$Z\left\{A(\sigma)\beta_j^{\infty}(k)\right\}=0\rightarrow$$

$$\left(A_q z^{q}+A_{q-1}z^{q-1}+...+A_1 z+A_0\right)\left(x_{j,q+q_j-1}z^{-N}+...+x_{j,0}z^{-N+q+q_j-1}\right)=-\hat{\beta}(z)\rightarrow$$

where $\hat{\beta}(z)$ is the initial conditions vector and is equal to:

$$\hat{\beta}(z)=\begin{pmatrix} z^{-N} & z^{-N+1} & \cdots & z^{-N+q-1}\end{pmatrix}\begin{pmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{pmatrix}\begin{pmatrix} \beta_j^{\infty}(N+q) \\ \vdots \\ \beta_j^{\infty}(N+2) \\ \beta_j^{\infty}(N+1)\end{pmatrix}$$

The values $\beta_j^{\infty}(\mathrm{N+q}),.....\beta_j^{\infty}(\mathrm{N+1})$ are defined outside our given interval, but since

$$\mathrm{A}(\sigma)\beta(\mathrm{k})=0$$

$$\leftrightarrow A_q\beta(k+q)+A_{q-1}\beta(k+q-1)+...+A_1\beta(k+1)+A_0\beta(k)=0$$

for k=N-q+1,…,N we have:

$$\begin{pmatrix} A_0 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & A_0 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta_j^{\infty}(N-q+1) \\ \vdots \\ \beta_j^{\infty}(N) \\ \vdots \\ \beta_j^{\infty}(N+q) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

More analytically:

For $k = N - q + 1$ we have:

$$A_q\beta(N+1) + A_{q-1}\beta(N) + \dots + A_0\beta(N-q+1) = 0 \rightarrow$$
$$A_q\beta(N+1) = -\left(A_{q-1}\beta(N) + \dots + A_0\beta(N-q+1)\right)$$

For $k = N - q + 2$ we have

$$A_q\beta(N+2) + A_{q-1}\beta(N+1) = -\left(A_{q-2}\beta(N) + \dots + A_0\beta(N-q+2)\right)$$

Continuing in this fashion we eventually have that

$$\hat{\beta}(z) = \begin{pmatrix} z^{-N} & z^{-N+1} & \cdots & z^{-N+q-1} \end{pmatrix} \begin{pmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta(N+q) \\ \vdots \\ \beta(N+2) \\ \beta(N+1) \end{pmatrix} =$$

$$= -z^{-N}\begin{pmatrix} z^{q-1}I_r & \cdots & \cdots & I_r \end{pmatrix} \begin{pmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \vdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} \beta(N-q+1) \\ \vdots \\ \beta(N-1) \\ \beta(N) \end{pmatrix}$$

Now considering the left side of the equation:

$$\left(A_q z^q + A_{q-1}z^{q-1} + \dots + A_1 z + A_0\right)\left(x_{j,q+q_j-1}z^{-N} + \dots + x_{j,0}z^{-N+q+q_j-1}\right) = -\hat{\beta}(z) \rightarrow$$

$$z^{-N}\left(\underbrace{z^{q+q+q_j-1}I_r \quad \cdots \quad zI_r \quad I_r}_{q+q+q_j}\right)\left(\begin{array}{cccccccc} A_q & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ A_{q-1} & A_q & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_0 & A_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & A_0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & A_0 & A_1 & A_2 & \cdots & A_q \\ \hline 0 & 0 & \ddots & \ddots & A_0 & \ddots & \ddots & A_{q-1} \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & & A_0 \end{array}\right)\begin{pmatrix} x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,q+q_j-1} \end{pmatrix} = -\hat{\beta}(z)$$

The part above the line multiplied by $\begin{pmatrix} x_{j,0} \\ x_{j,1} \\ \vdots \\ x_{j,q+q_j-1} \end{pmatrix}$ equals the zero vector, as we know from (6).

So we end up with:

$$z^{-N}\left(z^{q-1}I_r \quad \cdots \quad zI_r \quad I_r\right)\begin{pmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q_j-1} \\ x_{j,q_j} \\ \vdots \\ x_{j,q+q_j-1} \end{pmatrix} = -\hat{\beta}(z) \Rightarrow$$

$$\Rightarrow z^{-N}\left(z^{q-1}I_r \quad \cdots \quad zI_r \quad I_r\right)\begin{pmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}\begin{pmatrix} x_{j,q_j-1} \\ x_{j,q_j} \\ \vdots \\ x_{j,q+q_j-1} \end{pmatrix} =$$

$$= z^{-N}\left(z^{q-1}I_r \quad \cdots \quad \cdots \quad I_r\right)\begin{pmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \vdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}\begin{pmatrix} \beta_j^{\infty}(N-q+1) \\ \vdots \\ \beta_j^{\infty}(N-1) \\ \beta_j^{\infty}(N) \end{pmatrix}$$

This equation holds true because

$$\beta_j^{\infty}(N-q+1) = 0+0+\ldots+x_{j,q_j-1}\delta(0)+0+..+0 = x_{j,q_j-1}$$
$$\vdots$$
$$\beta_j^{\infty}(N) = x_{j,q+q_j-1}\delta(N-\mathrm{N}) + x_{j,q+q_j-2}\underbrace{\delta(N-\mathrm{N}-1)}_{=0} \underset{=0}{+\ldots+} x_{j,0}\underbrace{\delta(N-\mathrm{N}-q-q_j+1)}_{=0} = x_{j,q+q_j-1}$$

which means that the left and right hand of this equation are equal, thus

$$Z\left\{A(\sigma)\beta_j^{\infty}(k)\right\} = 0 \rightarrow A(\sigma)\beta_j^{\infty}(k) = 0 \text{ so the vector}$$
$$\beta_j^{\infty}(k) = x_{j,q_j-1}\delta(N-k) + x_{j,q_j-2}\delta(N-1-k)+\ldots+x_{j,0}\delta(N-q-q_j+1-k)$$

is a solution of A(σ)β(k)=0.

□

This Theorem tells us that the problem of finding a system in the form of (1) that has as a solution the vector

$$\beta_j^{\infty}(k) = x_{j,q+q_j-1}\delta(N-k) + x_{j,q+q_j-2}\delta(N-(k+1))+\ldots+x_{j,0}\delta(N-(k+q_j-1))$$

is equivalent to the problem of finding a system of the form of (3) having as a solution the vector

$$\tilde{\beta}_j(k) = x_{j,0}\delta(k-q-q_j+1) + x_{j,1}\delta(k-q-q_j+2)+\cdots+x_{j,q_j-1}\delta(k-q)+\cdots+x_{j,q+q_j-1}\delta(k)\ .$$

However this problem can easily be solved from the results we already have. These facts give rise to the following:

Theorem 27: Let $\beta_j^{\infty}(k) = \sum_{w=0}^{q+q_j-1} x_{j,l}\delta(\mathrm{N}-w-k)$ where each $x_{j,l}$ is a vector in $\mathbb{C}^r$, $1 \le j \le w < r$.

Define

$$C_{\infty j} = \begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q_j-1} & x_{j,q_j} & x_{j,q_j+1} & \cdots & x_{j,q+q_j-1} \end{pmatrix}$$

where $j = 1,2,\ldots,l$ and let

$$C = \begin{pmatrix} C_1 & C_2 & \cdots & C_l \end{pmatrix} \in \mathbb{R}^{r\times\mu},\ J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_l \end{pmatrix} \in \mathbb{R}^{\mu\times\mu}$$

with J_i the Jordan block of order $\mu_j = q+q_j$ with eigenvalue 0 and $\mu = \sum_{j=1}^{l}\mu_j$. Let $a \neq 0$ be a complex number and define

$$\tilde{A}(\sigma) = I_r - C\left(J - aI_r\right)^{-q}\left\{(\sigma - a)V_q + (\sigma - a)^2 V_{q-1} + \cdots + (\sigma - a)^q V_1\right\}$$

where q=ind(C,J) is the least integer such that the matrix

$S_{q-1} = \begin{pmatrix} C \\ CJ \\ \vdots \\ CJ^{q-1} \end{pmatrix}$ has full column rank and $V = \begin{pmatrix} V_1 & V_2 & \cdots & V_q \end{pmatrix}$ is the left inverse of

$$S_{1-q} = \begin{pmatrix} C \\ C\left(J - aI_r\right)^{-1} \\ \vdots \\ C\left(J - aI_r\right)^{1-q} \end{pmatrix} \text{ i.e. } VS_{1-q} = I_{rq}\,.$$

Then $\beta_j^{\infty}(k)$ j=1,,2,…,l are solutions of the equation A(σ)β(k)=0 where $A(\sigma) = \sigma^q \tilde{A}(\frac{1}{\sigma})$.

Furthermore, q is the minimal possible degree of any $r \times r$ matrix polynomial with this property.

□

Example 28: Suppose that we want to find out a polynomial matrix A(σ) such that the AR-representation has the following solution

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{x_{1,2}} \delta(\mathrm{N} - k) + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{x_{1,1}} \delta(N - k - 1) + \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{x_{1,0}} \delta\left(N - (k + 2)\right)$$

which means we have $q + q_1 = 3$. Assume first that q=1. Then we can construct the matrices

$$C = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix C has det(C)=-2 . So q=ind(C,J)=1

$$V_q = V_1 = C^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Let a=1. We have

$$\tilde{A}(\sigma) = I_3 - C\left(J - I_3\right)^{-1}(s-1)V_1 =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^{-1} (\sigma-1) \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2\sigma-1 & \sigma-1 & \sigma-1 \\ \frac{1}{2}-\frac{1}{2}\sigma & 1 & \frac{1}{2}-\frac{1}{2}\sigma \\ 0 & \sigma-1 & \sigma \end{pmatrix}$$

with Smith form $S^{0}_{\tilde{A}(\sigma)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$ as expected.

So the matrix we are looking for is $A(\sigma) = \sigma\tilde{A}(\frac{1}{\sigma}) = \begin{pmatrix} 2-\sigma & 1-\sigma & 1-\sigma \\ \frac{1}{2}\sigma-\frac{1}{2} & \sigma & \frac{1}{2}\sigma-\frac{1}{2} \\ 0 & 1-\sigma & 1 \end{pmatrix}$

with

$$S^{\infty}_{A(\sigma)} = \sigma S^{0}_{\tilde{A}(\sigma)}(\frac{1}{\sigma}) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \frac{1}{\sigma^2} \end{pmatrix}$$

and

$$A_0CJ + A_1C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; rank\begin{pmatrix} C \\ CJ \\ CJ^2 \end{pmatrix} = 3$$

□

Example 29: We want to find the polynomial matrix A(σ) with given backward behavior

$$\beta_1(k) = +\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{x_{1,3}}\delta(N-k) + \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{x_{1,2}}\delta(N-k-1) + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_{1,1}}\delta(N-k-2) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{1,0}}\delta\left(N-k-3\right)$$

which means we have $q+q_1=4$. We can start by assuming that q=1. Then we define the matrices

$$C = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $S_0 = C$ has not full column rank, so the assumption q=1 is dismissed. Now assume that q=2.

$$S_{2-1} = S_1 = \begin{pmatrix} C \\ CJ \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

this matrix has full column rank because $\det(S_1) \neq 0$. Therefore, we have that q=ind(C,J)=2.

Let also a=1 and define

$$\tilde{A}(\sigma) = I_2 - C\left(J - I_4\right)^{-2}\left\{(\sigma-1)V_2 + (\sigma-\alpha)^2 V_1\right\}$$

$$\text{where } V = \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} C \\ C\left(J - I_4\right)^{-1} \end{pmatrix}^{-1}.$$

$$\text{we have } C\left(J - I_4\right)^{-1} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -2 & -1 & -1 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$V = \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & -2 & -1 & -1 \\ 0 & -1 & -1 & -2 \end{pmatrix}^{-1} = \left(\begin{array}{cc|cc} 1 & -2 & 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -1 \end{array}\right)$$

$$\tilde{A}(\sigma)=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}1&1&-1&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\\0&0&0&-1\end{pmatrix}^{-2}\left\{(\sigma-1)\begin{pmatrix}0&1\\-\frac{1}{2}&1\\-\frac{1}{2}&0\\\frac{1}{2}&-1\end{pmatrix}+(\sigma-1)^2\begin{pmatrix}1&-2\\-\frac{1}{2}&\frac{3}{2}\\-\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\right\}$$

$$=\begin{pmatrix}\sigma&2\sigma^2-\sigma-1\\\frac{1}{2}\sigma-\frac{1}{2}\sigma^2&\frac{3}{2}\sigma^2-\frac{1}{2}\end{pmatrix}$$

with Smith form

$$S^0_{\tilde{A}(\sigma)}=\begin{pmatrix}1&0\\0&\sigma^4\end{pmatrix}$$

The dual matrix that we are looking for is

$$A(\sigma)=\sigma^2\tilde{A}(\frac{1}{\sigma})=\begin{pmatrix}\sigma&-\sigma^2-\sigma+2\\\frac{1}{2}\sigma-\frac{1}{2}&-\frac{1}{2}\sigma^2+\frac{3}{2}\end{pmatrix}$$

with $\det\left[A(\sigma)\right]=1$ and Smith form at infinity

$$S^\infty_{A(\sigma)}(\sigma)=\sigma^2 S^0_{\tilde{A}(\sigma)}(\frac{1}{\sigma})\begin{pmatrix}\sigma^2&0\\0&\frac{1}{\sigma^2}\end{pmatrix}$$

and

$$A_0CJ^2+A_1CJ+A_2C=\begin{pmatrix}0&0&0&0\\0&0&0&0\end{pmatrix}\ ;\ rank\begin{pmatrix}C\\CJ\\CJ^2\\CJ^3\end{pmatrix}=4$$

□

CHAPTER 4

CONSTRUCTION OF A SYSTEM OF ALGEBRAIC DIFFERENCE EQUATIONS WITH GIVEN FORWARD AND BACKWARD SOLUTION SPACE

In Chapter 3, we studied the construction of a system in the form of an AR-representation that satisfied a given forward behavior. A theorem was also provided for the case of backward behavior. We shall now combine these two results, in order to create a method for constructing a system that satisfies both a forward and a backward behavior of our choice.

Theorem 30: Let $A(\sigma)=A_q\sigma^q+A_{q-1}\sigma^{q-1}+...+A_0\in\mathbb{R}^{r\times r}[\sigma]$ with $rank_{R(\sigma)}A(\sigma)=r$ and $\sigma_j\neq 0$ such that $\det(A(\sigma_j))=0$ (σ_j is a zero of A(σ)). If $\beta_j(k)=C_jJ_j^kx_0$ (where $\left(C_j\in\mathbb{R}^{r\times n_j},J_j\in\mathbb{R}^{n_j\times n_j}\right)$ is a finite Jordan pair corresponding to the zero σ_j of $A(\sigma)$) is a solution of the AR-representation A(σ)β(k)=0, then $\tilde{\beta}_j(k)=C_jJ_j^{-1}\left(J_j^{-1}\right)^kx_0$ is a solution of the dual representation $\tilde{\mathrm{A}}(\sigma)\tilde{\beta}(k)=0$.

Proof: Since $\beta(k)=C_jJ_j^kx_0$ is a solution of A(σ)β(k)=0, we have that

$$\begin{pmatrix}\beta(0)\\ \beta(1)\\ \vdots\\ \beta(q-1)\end{pmatrix}=\begin{pmatrix}C_j\\ C_jJ_j\\ \vdots\\ C_jJ_j^{q-1}\end{pmatrix}x_0$$

Taking Z transform in A(σ)β(k)=0 we have that

$$A(z)C_j(zI_{n_j})\left(zI_{n_j}-J_j\right)^{-1}=\begin{pmatrix}z^qI_r & z^{q-1}I_r & \cdots & zI_r\end{pmatrix}\begin{pmatrix}A_q & 0 & \cdots & 0\\ A_{q-1} & A_q & \cdots & \vdots\\ \vdots & \vdots & \ddots & 0\\ A_1 & A_2 & \cdots & A_q\end{pmatrix}\begin{pmatrix}C_j\\ C_jJ_j\\ \vdots\\ C_jJ_j^{q-1}\end{pmatrix}x_0$$

replacing z with $\frac{1}{z}$ and multiplying both sides by z^q

$$\tilde{\mathrm{A}}(z)C_j(\frac{1}{z}I_{n_j})\left(\frac{1}{z}I_{n_j}-J_j\right)^{-1}x_0=\begin{pmatrix}I_r & zI_r & \cdots & z^{q-1}I_r\end{pmatrix}\begin{pmatrix}A_q & 0 & \cdots & 0\\ A_{q-1} & A_q & \cdots & \vdots\\ \vdots & \vdots & \ddots & 0\\ A_1 & A_2 & \cdots & A_q\end{pmatrix}\begin{pmatrix}C_j\\ C_jJ_j\\ \vdots\\ C_jJ_j^{q-1}\end{pmatrix}x_0\Rightarrow$$

$$\to \tilde{\mathrm{A}}(z)C_j\left(I_{n_j}-zJ_j\right)^{-1}x_0=\begin{pmatrix} z^{q-1}I_r & z^{q-1}I_r & \cdots & I_r\end{pmatrix}\begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ A_2 & A_3 & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ A_q & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1}\end{pmatrix}x_0\Rightarrow$$

The matrix J_j is invertible so therefore

$$\to \tilde{\mathrm{A}}(z)C_j\left(\left(J_j^{-1}-zI_{n_j}\right)J_j\right)^{-1}x_0=\begin{pmatrix} z^{q-1}I_r & z^{q-1}I_r & \cdots & I_r\end{pmatrix}\begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ A_2 & A_3 & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ A_q & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix} C_j \\ C_jJ_j \\ \vdots \\ C_jJ_j^{q-1}\end{pmatrix}x_0\Rightarrow$$

$$\to -\tilde{\mathrm{A}}(z)C_jJ_j^{-1}\left(zI_{n_j}-J_j^{-1}\right)^{-1}x_0=\begin{pmatrix} z^{q-1}I_r & z^{q-1}I_r & \cdots & I_r\end{pmatrix}\begin{pmatrix} A_1 & A_2 & \cdots & A_q \\ A_2 & A_3 & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ A_q & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix} C_jJ_j \\ C_jJ_j^2 \\ \vdots \\ C_jJ_j^{q}\end{pmatrix}J_j^{-1}x_0\Rightarrow$$

Since the pair $\left(C_j, J_j\right)$ is a finite Jordan pair of A(σ) we have that

$$A_0C_j+A_1C_jJ_j+\cdots+A_qC_jJ_j^q=0\text{, So}$$

$$\xrightarrow{Z} -\tilde{\mathrm{A}}(z)C_jJ_j^{-1}\left(zI_{n_j}\right)\left(zI_{n_j}-J_j^{-1}\right)^{-1}x_0=\begin{pmatrix} z^{q}I_r & z^{q-1}I_r & \cdots & zI_r\end{pmatrix}\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0\end{pmatrix}\begin{pmatrix} -C_j \\ -C_jJ_j^{-1} \\ \vdots \\ -C_jJ_j^{-q+1}\end{pmatrix}J_j^{-1}x_0$$

$$\Rightarrow \tilde{\mathrm{A}}(z)C_jJ_j^{-1}\left(zI_{n_j}\right)\left(zI_{n_j}-J_j^{-1}\right)^{-1}x_0=\begin{pmatrix} z^{q}I_r & z^{q-1}I_r & \cdots & zI_r\end{pmatrix}\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0\end{pmatrix}\begin{pmatrix} C_jJ_j^{-1} \\ C_jJ_j^{-2} \\ \vdots \\ C_jJ_j^{-q}\end{pmatrix}x_0$$

Therefore we have concluded that

$$\tilde{\beta}(k)=\mathrm{Z}^{-1}\left\{\tilde{\beta}(z)\right\}=\mathrm{Z}^{-1}\left\{C_jJ_j^{-1}\left(zI_{n_j}\right)\left(zI_{n_j}-J_j^{-1}\right)^{-1}x_0\right\}=C_jJ_j^{-1}\left(J_j^{-1}\right)^k x_0$$

is a solution of the dual AR-representation $\tilde{\mathrm{A}}(\sigma)\tilde{\beta}(k)=0$ for initial conditions

$$\begin{pmatrix} \tilde{\beta}(0) \\ \tilde{\beta}(1) \\ \vdots \\ \tilde{\beta}(q-1) \end{pmatrix} = \begin{pmatrix} C_j J_j^{-1} \\ C_j J_j^{-2} \\ \vdots \\ C_j J_j^{-q} \end{pmatrix} x_0$$

□

Since the matrix J_j^{-1} is not in Jordan form, we can find a nonsingular constant matrix $U \in \mathbb{R}^{n_j \times n_j}$ such that $J_j^{-1} = U\tilde{J}_j U^{-1}$ where $\tilde{J}_j$ is in Jordan form. With this change, the solution of $\tilde{\mathrm{A}}(\sigma)\tilde{\beta}(k) = 0$ can also be written as follows:

$$\tilde{\beta}(k) = C_j J_j^{-1}\left(J_j^{-1}\right)^k x_0 = C_j U\tilde{J}_j U^{-1} U\left(\tilde{J}_j\right)^k U^{-1} x_0 = \tilde{C}_j\left(\tilde{J}_j\right)^k\left(U^{-1}x_0\right)$$

where $\tilde{C}_j = C_j U\tilde{J}_j$ and we used the fact that

$$\left(U\tilde{J}_j U^{-1}\right)^k = U\tilde{J}_j \underbrace{U^{-1}\cdot U}_{I_{n_j}} \tilde{J}_j \underbrace{U^{-1}\cdot U}_{I_{n_j}} \tilde{J}_j U^{-1} \cdots U\tilde{J}_j U^{-1} = U\left(\tilde{J}_j\right)^k U^{-1}$$

So we can see that instead of using the matrix pair $\left(C_j J_j^{-1} \in \mathbb{R}^{r\times n_j}, J_j^{-1} \in \mathbb{R}^{n_j \times n_j}\right)$ where the matrix J_j^{-1} is not in Jordan form, we can use the matrix pair

$$\left(\tilde{C}_j = C_j U\tilde{J}_j \in \mathbb{R}^{r\times n_j}, \tilde{J}_j = U^{-1}J_j^{-1}U \in \mathbb{R}^{n_j\times n_j}\right)$$

where $\tilde{J}_j$ is in Jordan form.

Summarizing these results, in order to construct an AR-representation for a certain forward and backward behavior, one must follow the next algorithm.

Algorithm 31: Construction of an AR-representation with given forward & backward behavior. (except from polynomial behavior).

Step 1: Transform the finite Jordan pairs $\left(C_j \in \mathbb{R}^{r\times n_j}, J_j \in \mathbb{R}^{n_j\times n_j}\right)$ that correspond to solutions of the form $\beta(k) = C_j J_j^k x_0$ to the finite Jordan pairs $\left(\tilde{C}_j = C_j U\tilde{J}_j \in \mathbb{R}^{r\times n_j}, \tilde{J}_j = U^{-1}J_j^{-1}U \in \mathbb{R}^{n_j\times n_j}\right)$ that correspond to the solutions of the form $\tilde{\beta}(k) = \tilde{C}_j\left(\tilde{J}_j\right)^k\left(U^{-1}x_0\right)$ of the dual system that we are looking for.

Step 2: Create the infinite Jordan pairs that correspond to solutions of the form (6). These are also the finite Jordan pairs for the f.e.d. at zero of the dual system.

Step 3: Construct the polynomial matrix $\tilde{A}(\sigma)$, using the method presented in Chapter 2

Step 4: Get the polynomial matrix $A(\sigma)=\sigma^{q}A(\frac{1}{\sigma})$ that we are looking for and thus the AR-representation is A(σ)β(k)=0. □

Algorithm 31 can be implemented in Mathematica.

Algorithm 31

```
c1=Input["input matrix C, corresponding to the Finite Jordan Pair"]
j1=Input["input matrix J, corresponding to the Finite Jordan Pair"]
{u,jnew}=JordanDecomposition[Inverse[j1]];
cnew=c1.u.jnew;
cinf=Input["Input matrix C.inf, corresponding to the Infinite Jordan
Pair"];
jinf=Input["Input matrix J.inf, corresponding to the Infinite Jordan
Pair"];
Cpair=ArrayFlatten[{{cnew,cinf}}];
{rowsj1,columnsj1}=Dimensions[j1];
{rowsjinf,columnsjinf}=Dimensions[jinf];
Jpair=ArrayFlatten[{{jnew,ConstantArray[0,{rowsj1,columnsjinf}]},{Cons
tantArray[0,{rowsjinf,columnsj1}],jinf}}];
q=1;
{rowsC,columnsC}=Dimensions[Cpair];
S=Cpair;
While[MatrixRank[S]<columnsC,
 q=q+1;
 S=ArrayFlatten[{{S},{Cpair.MatrixPower[Jpair,q-1]}}]]
a=Input["Choose a value a, other than the f.e.d.'s of the desired
system"];
M=Cpair;
For[i=1,i<q,i=i+1,
 M=ArrayFlatten[{{M},{Cpair.MatrixPower[Jpair-a
IdentityMatrix[Dimensions[Jpair]],-i]}}]]
M=PseudoInverse[M];
{rowsM,columnsM}=Dimensions[M];
step=columnsM/q;
indicator=1
For[i=1, i<=q, i=i+1,
 v[i]=Take[M,{1,rowsM},{indicator,indicator+step-
1}];indicator=indicator+step]
sum=(s-a)^q v[1]
For[i=1,i<q,i++,sum=sum+(s-a)^(q-i) v[i+1]]
sum=sum//Simplify;
A=IdentityMatrix[rowsC]-Cpair.MatrixPower[Jpair-a
IdentityMatrix[Dimensions[Jpair]],-q].sum//Simplify;
A=A/.s→1/s;
A=A s^q
```

Example 32: We want to find an AR-representation $A(\sigma)\beta(k)=0$ with the following forward and backward behavior

$$\beta_1(k)=\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\beta_{1,1}}2^k+\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\beta'_{1,0}}k2^k=\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\beta_{1,1}}2^k+\underbrace{\begin{pmatrix}2\\0\end{pmatrix}}_{\beta_{1,0}}k2^{k-1}\text{ and}$$

$$\beta_2(k)=\underbrace{\begin{pmatrix}-1\\-1\end{pmatrix}}_{x_{1,2}}\delta(\mathrm{N}-k)+\underbrace{\begin{pmatrix}-1\\1\end{pmatrix}}_{x_{1,1}}\delta(N-k+1)+\underbrace{\begin{pmatrix}0\\3\end{pmatrix}}_{x_{1,0}}\delta(N-k+2)$$

We first define the matrix pair

$$C_1=\begin{pmatrix}\beta_{1,0} & \beta_{1,1}\end{pmatrix}=\begin{pmatrix}2 & 1\\0 & -1\end{pmatrix}\text{ and } J_1=\begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}$$

$$J_1^{-1}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{4}\\0 & \frac{1}{2}\end{pmatrix}=\underbrace{\begin{pmatrix}1 & 0\\0 & -4\end{pmatrix}}_{U}\underbrace{\begin{pmatrix}\frac{1}{2} & 1\\0 & \frac{1}{2}\end{pmatrix}}_{\tilde{J}_1}\underbrace{\begin{pmatrix}1 & 0\\0 & -4\end{pmatrix}}_{U^{-1}}$$

$$\tilde{C}_1=C_1U\tilde{J}_1=\begin{pmatrix}2 & 1\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -4\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 1\\0 & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$$

$$C_2=\begin{pmatrix}x_{1,0} & x_{1,1}\end{pmatrix}=\begin{pmatrix}0 & -1 & -1\\3 & 1 & -1\end{pmatrix},\ J_2=\begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$$

The whole matrix pair is

$$C=\begin{pmatrix}\tilde{C}_1 & C_2\end{pmatrix}=\left(\begin{array}{cc|ccc}1 & 0 & 0 & -1 & -1\\0 & 2 & 3 & 1 & -1\end{array}\right)$$

$$J=\begin{pmatrix}J_1 & 0\\0 & J_2\end{pmatrix}=\left(\begin{array}{cc|ccc}\frac{1}{2} & 1 & 0 & 0 & 0\\0 & \frac{1}{2} & 0 & 0 & 0\\\hline 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{array}\right)$$

Now we start assuming different values for q. First assume that q=1, in this case we have that the matrix

$C = \begin{pmatrix} \tilde{C}_1 & C_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 2 & 3 & 1 & -1 \end{pmatrix}$ does not have full column rank, so $q \neq 1$

Now assume that q=2

$$S_2 = \begin{pmatrix} C \\ CJ \end{pmatrix} = \left(\begin{array}{ccccc} 1 & 0 & 0 & -1 & -1 \\ 0 & 2 & 3 & 1 & -1 \\ \hline \frac{1}{2} & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 & 1 \end{array}\right)$$

Because this matrix does not have full column rank, $q \neq 2$.

Now assume that q=3.

$$S_3 = \begin{pmatrix} C \\ CJ \\ CJ^2 \end{pmatrix} = \left(\begin{array}{ccccc} 1 & 0 & 0 & -1 & -1 \\ 0 & 2 & 3 & 1 & -1 \\ \hline \frac{1}{2} & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ \hline \frac{1}{4} & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 3 \end{array}\right)$$

The matrix S_3 has full column rank, so q=3.

Let a=1. We have

$$\tilde{A}(\sigma) = I_2 - C\left(J - I_4\right)^{-3}\left\{(s-1)V_3 + (s-1)^2 V_2 + (s-1)^3 V_1\right\}$$

$$\text{where } (V_1 \quad V_2 \quad V_3) = \begin{pmatrix} C \\ C\left(J - I_8\right)^{-1} \\ C\left(J - I_8\right)^{-2} \end{pmatrix}^{-1} =$$

$$\left(\begin{array}{cc|cc|cc} \frac{5731}{11623} & -\frac{3910}{11623} & -\frac{7606}{11623} & -\frac{4151}{11623} & -\frac{2330}{11623} & -\frac{241}{11623} \\ -\frac{1227}{11623} & \frac{2017}{23246} & \frac{3555}{23246} & \frac{1414}{11623} & \frac{2391}{23246} & \frac{811}{23246} \\ \frac{10798}{34869} & \frac{6568}{11623} & \frac{2545}{34869} & \frac{14563}{34869} & -\frac{1427}{34869} & \frac{6482}{34869} \\ -\frac{4194}{11623} & -\frac{7778}{11623} & -\frac{3757}{11623} & -\frac{12104}{11623} & -\frac{830}{11623} & -\frac{4326}{11623} \\ \frac{2926}{11623} & \frac{2644}{11623} & \frac{1047}{11623} & \frac{5845}{11623} & -\frac{208}{11623} & \frac{3201}{11623} \end{array}\right)$$

So we have

$$\tilde{A}(\sigma) = I_2 - C\left(J - I_4\right)^{-3}\left\{(s-1)V_3 + (s-1)^2 V_2 + (s-1)^3 V_1\right\} =$$

$$\begin{pmatrix} \frac{15816 - 69350s + 85715s^2 - 20558s^3}{11623} & \frac{2s\left(2636 - 9801s + 7165s^2\right)}{11623} \\ \frac{3\left(456 - 77s - 2961s^2 + 2582s^3\right)}{11623} & \frac{s\left(456 + 227s + 10940s^2\right)}{11623} \end{pmatrix}$$

Therefore, the polynomial matrix we are looking for is (we multiply by 11623 to get a simpler result)

$$A(\sigma) = 11623\sigma^3 \tilde{A}(\frac{1}{\sigma}) =$$

$$\begin{pmatrix} 15816s^3 - 69350s^2 + 85715 - 20558 & 2\left(2636s^2 - 9801s + 7165\right) \\ 3\left(456s^3 - 77s^2 - 2961s + 2582\right) & \left(456s^2 + 227s + 10940\right) \end{pmatrix}$$

and by multiplying by 11623 we get

And as a matter of fact, this matrix A(σ) has the pairs $\left(C_1 \quad J_1\right)$ and $\left(C_2 \quad J_2\right)$ as Jordan pairs, which is easily checked in Mathematica.

The whole string of commands In Mathematica is

```
c={{1,0,0,-1,-1},{0,2,3,1,-1}}
j={{1/2,1,0,0,0},{0,1/2,0,0,0},{0,0,0,1,0},{0,0,0,0,1},{0,0,0,0,0}}
b=1
ArrayFlatten[{{c},{c.j}}]
MatrixRank[ArrayFlatten[{{c},{c.j}}]]
(rank equals 3 so we move setting q=3)
MatrixRank[ArrayFlatten[{{c},{c.j},{c.MatrixPower[j,2]}}]]
(result equals 4, so q=3)
m=ArrayFlatten[{{c},{c.MatrixPower[j-b IdentityMatrix[5],-
1]},{c.MatrixPower[j-b IdentityMatrix[4],-2]}}]
m1=PseudoInverse[m]
v1=Take[m1,{1,4},{1,2}]
v2=Take[m1,{1,4},{3,4}]
v3=Take[m1,{1,4},{5,6}]
a=IdentityMatrix[2]-c.MatrixPower[j-b IdentityMatrix[5],-3].((s-
b)v3+(s-b)^2 v2+(s-b)^3 v1)//Simplify
ad=a/.{s->1/s}
adual=s^3 abar//Simplify
a0=adual/.s->0
a1=(adual-a0)*(1/s)//Expand
a1=%/.s->0
a2={{-69350/11623,2 2636/11623},{3 (-77)/11623,456/11623}}
a3={{15816/11623,0},{3 (456/11623),0}}
c1={{1,0},{0,2}}
j1={{1/2,1},{0,1/2}}
a3.c1.MatrixPower[j1,3]+a2.c1.MatrixPower[j1,2]+a1.c1.MatrixPower[j1,1]
+a0.c1
Output: {{0,0},{0,0}}
(First Jordan Chain Checked)
c2={{0,-1,-1},{3,1,-1}}
j2={{0,1,0},{0,0,1},{0,0,0}}
a0.c2.MatrixPower[j2,3]+a1.c2.MatrixPower[j2,2]+a2.c2.MatrixPower[j2,1]
+a3.c2
{{0,0,0},{0,0,0}}
(Second Jordan Chain Checked)
```

We have constructed an algorithm of finding the polynomial matrix A(σ) corresponding to certain behavior. As we have already mentioned, A(σ) is not the only polynomial matrix satisfying the given behavior. Any other unimodular equivalent matrix $\hat{A}(s)$ such that $\hat{A}(s) = U(\sigma)\tilde{A}(\sigma)$ with $\det U = c \in \mathbb{R}$ gives a solution to our problem. More specifically, all the *divisor equivalent* polynomial matrices are solutions to our problem.

Definition 33 [Karampetakis et al 2004]: $A_1(\sigma), A_2(\sigma) \in \mathbb{R}[\sigma]^{r\times m}$ are said to be *divisor equivalent* if there exist polynomial matrices $M(s), N(s)$ such that:

$$\begin{pmatrix} M(s) & A_2(s) \end{pmatrix} \begin{pmatrix} A_1(s) \\ -N(s) \end{pmatrix} = 0$$

and the compound matrices $\begin{pmatrix} M(s) & A_2(s) \end{pmatrix}$ and $\begin{pmatrix} A_1(s) \\ -N(s) \end{pmatrix}$ satisfy:

(i) have full rank over $\mathbb{R}[s]$ and no f.e.ds.

(ii) have no i.e.ds.

□

Divisor Equivalence is very important for us, since it has the property of preserving both the finite and the infinite elementary divisors of a polynomial matrix.

In Theorem 28 and Algorithm 29 we used the transformation $z \to \frac{1}{z}$. As a result, the algorithm works well only in the case where there are no polynomial functions in the behavior space and the matrix J is invertible. For example, if the matrix J corresponds to a f.e.d. at zero, its diagonal elements would be zero, thus J won't be invertible. We can overcome this problem by using the transformation $z \to \frac{1}{z} + b$ and thus moving all possible zeros of A(σ) to non-zero places. We do this by following these remarks

- If (C_1, J_1) is a finite Jordan pair of A(σ), then $(C_1, J_1 + bI_n)$ is a finite Jordan pair of A(σ-β).

- If (C_2, J_2) is an infinite Jordan pair of A(σ), then $\left(C_2, J_2\left(I_n + bJ_2\right)^{-1}\right)$ is an infinite Jordan pair of A(σ-b).

Example 34: We want to find an AR-representation A(σ)β(k)=0 with the following forward and backward behavior

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\beta_{1,1}} \delta(k) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\beta_{1,0}} \delta(k-1) \text{ and } \beta_2(k) = \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(\mathrm{N}-k) + \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{x_{1,0}} \delta(N-k+1)$$

We define the matrix pair $C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $J^*{}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

but we are not going to use this Jordan pair. Instead we will create the new pair

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \text{ and } J_1 = J_1^* + 2I_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

that will give us the dual polynomial matrix of A(σ-1). Let also

$$J_1^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\tilde{J}_1} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}}_{U^{-1}}$$

$$\tilde{C}_1 = C_1 U \tilde{J}_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 2 \end{pmatrix}$$

The infinite Jordan pair is

$$\mathrm{C}_2 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}; \ \mathrm{J}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

but instead we are using the pair $\left(C_2 \quad J_2\left(I_2 + J_2\right)^{-1}\right)$

$$J_2\left(I_2 + 2J_2\right)^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let

$$\begin{pmatrix} \tilde{C}_1 & C_2 \end{pmatrix} = \left(\begin{array}{cc|cc} \frac{1}{2} & -1 & -1 & -1 \\ 0 & 2 & 1 & -1 \end{array}\right)$$

$$J = \begin{pmatrix} \tilde{J}_1 & 0 \\ 0 & J_2 \end{pmatrix} = \left(\begin{array}{cc|cc} \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now we start assuming values for q. For q=1 the matrix C obviously does not have full column rank.

For q=2

$$S_2 = \begin{pmatrix} C \\ CJ \end{pmatrix} = \left(\begin{array}{cccc} \frac{1}{2} & -1 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ \hline \frac{1}{4} & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

with $\det(S_2) = \frac{1}{4} \neq 0$, and so q=2 is accepted.

Let a=2. We have

$$\tilde{B}(\sigma) = I_2 - C\left(J - 2I_4\right)^{-2}\left\{(\sigma-2)V_2 + (\sigma-2)^2 V_1\right\}$$

where $\begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} C \\ C(J-2I)^{-1} \end{pmatrix}^{-1} = \left(\begin{array}{cc|cc} -20 & -6 & -33 & -5 \\ 3 & 0 & \frac{9}{2} & -\frac{3}{2} \\ -10 & -1 & -15 & 1 \\ -4 & -2 & -6 & -2 \end{array}\right)$

Therefore

$$\tilde{B}(\sigma) = I_2 - C\left(J - 2I_4\right)^{-2}\left\{(\sigma-2)V_2 + (\sigma-2)^2 V_1\right\} = \begin{pmatrix} \frac{1}{36}\left(2-11\sigma+14\sigma^2\right) & \frac{1}{36}\left(2-7\sigma+3\sigma^2\right) \\ \frac{1}{12}\left(-2+5\sigma-2\sigma^2\right) & \frac{1}{12}\left(-2+\sigma+3\sigma^2\right) \end{pmatrix}$$

The dual polynomial matrix A(σ-2) is

$$\mathrm{A}(\sigma-2) = \sigma^2 \tilde{B}(\frac{1}{\sigma}) = \begin{pmatrix} \frac{1}{36}\left(2\sigma^2-11\sigma+14\right) & \frac{1}{36}\left(2\sigma^2-7\sigma+3\right) \\ \frac{1}{12}\left(-2+5\sigma-2\sigma^2\right) & \frac{1}{12}\left(-2\sigma^2+\sigma+3\right) \end{pmatrix}$$

So

$$A(\sigma) = \begin{pmatrix} \frac{1}{36}(2\sigma^2-3\sigma) & \frac{1}{36}(2\sigma^2+\sigma-3) \\ -\frac{1}{12}(2\sigma^2+3\sigma) & -\frac{1}{12}(2\sigma^2+7\sigma+3) \end{pmatrix}$$

with Smith form

$$S^{\mathbb{C}}_{\mathrm{A}(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

and Smith form at infinity

$$S^{\infty}_{\mathrm{A}(\sigma)}(\sigma) = \sigma^2 S^{0}_{\tilde{\mathrm{A}}(\sigma)}(\frac{1}{\sigma}) = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Both Smith forms are exactly what we expected and confirm the correctness of our algorithm.

Again we check the initial Jordan pairs:

$$A_0 C_1 + A_1 C_1 J^*_1 + A_2 C_1 J^{*2}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \;;\; rank\begin{pmatrix} C_1 \\ C_1 J^*_1 \\ C_1 J^{*2}_1 \end{pmatrix} = 2$$

$$A_2 C_2 + A_1 C_2 J_2 + A_0 C_2 J_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \;;\; rank\begin{pmatrix} C_2 \\ C_2 J_2 \\ C_2 J_2^2 \end{pmatrix} = 2$$

□

CHAPTER 5

Construction Of A System Of Algebraic Difference Equations With Given Forward & Backward Behavior Via The Decomposable Pairs Method.

In this chapter we will present an alternative method of constructing a system of difference equations first presented by Gohberg. This method uses the theory of decomposable pairs of a matrix A(σ).

Definition 35 [Gohberg et al 1982]: A matrix pair (X,T) is called *decomposable* of order p if $X \in \mathbb{R}^{r\times p}$ and $T \in \mathbb{R}^{p\times p}$. An admissible pair (X,T) of order rq is called a decomposable pair of degree q if the following conditions are satisfied:

i) $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$; $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$

where $X_1 \in \mathbb{R}^{r\times n}, T_1 \in \mathbb{R}^{n\times n}$ for some $0 \le n \le rq$, so that $X_2 \in \mathbb{R}^{r\times(rq-n)}; T_2 \in \mathbb{R}^{(rq-n)\times(rq-n)}$.

ii) The matrix

$$S_{q-1} = \begin{pmatrix} X_1 & X_2 T_2^{q-1} \\ X_1 T_1 & X_2 T_2^{q-2} \\ \vdots & \vdots \\ X_1 T_1^{q-1} & X_2 \end{pmatrix}$$

is nonsingular.

A decomposable pair (X,T) will be called a decomposable pair of the regular polynomial matrix A(σ) defined in (1) if in addition to (i) and (ii) the following condition holds

iii) $\sum_{i=0}^{q} A_i X_1 T_1^i = 0$; $\sum_{i=0}^{q} A_i X_2 T_2^{q-i} = 0$.

□

A decomposable pair appears to include the full spectral information of a polynomial matrix. That is both the finite and infinite structure.

The existence of a decomposable pair for every nonregular polynomial matrix follows from the following theorem.

Theorem 36 [Gohberg et al 1982]: Let A(σ) defined in (1) and $\begin{pmatrix} C & J \end{pmatrix}$ and $\begin{pmatrix} C_\infty & J_\infty \end{pmatrix}$ be its finite and infinite Jordan Pairs. Then

$$X = \begin{pmatrix} C & C_\infty \end{pmatrix} \;;\; T = \begin{pmatrix} J & 0 \\ 0 & J_\infty \end{pmatrix}$$

is a decomposable pair of A(σ).

□

A very important result is that we can always construct a regular polynomial matrix A(σ) corresponding to a given decomposable pair (X,T). The way is given in the following theorem

Theorem 37 [Gohberg et al 1982]: Let (X,T) be the decomposable pair of degree q given in definition 35 and let S_{q-1} be the matrix defined in the same def. Then for every r x rq matrix V such that the matrix $\begin{pmatrix} S_{q-2} \\ V \end{pmatrix}$ has full column rank, the polynomial matrix

$$A(\sigma) = V\left(I-P\right)\begin{pmatrix} \sigma I - \mathrm{T}_1 & 0 \\ 0 & \sigma T_2 - I \end{pmatrix}\left(U_0 + U_1\sigma + \ldots + U_{q-1}\sigma^{q-1}\right)$$

has (X,T) as its decomposable pair, where

$$P = \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} S_{q-1}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} S_{q-2}$$

with

$$S_{q-2} = \begin{pmatrix} X_1 & X_2 T_2^{q-2} \\ X_1 T_1 & X_2 T_2^{q-3} \\ \vdots & \vdots \\ X_1 T_1^{q-2} & X_2 \end{pmatrix}$$

and

$$\begin{pmatrix} U_0 & U_1 & \cdots & U_{q-1} \end{pmatrix} = S_{q-1}^{-1}$$

□

Therefore, given a decomposable pair (X,T) of degree q, we can always create a polynomial matrix A(σ) of degree q. Using the above theorem we construct the following algorithm.

Algorithm 38: Construction of a system of algebraic difference equations with given forward & backward behavior.

Suppose that we are given the vector space

$$B_F^D = \left\langle \xi_{j,q}^i(k) := \lambda_i^k x_{j,q}^i + k\lambda_i^{k-1} x_{j,q-1}^i + \ldots + \binom{k}{q} \lambda_i^{k-q} x_{j,0}^i \right\rangle$$

$i \in k$; j=z,z+1,…,r; $q = 0,1,...,\sigma_{i,j-1}$ and the vector space

$$B_B^D = \left\langle \xi_{j,i}^B(k) := x_{j,i}\delta(N-k) + x_{j,i-1}\delta(N-(k+1)) + \ldots + x_{j,0}\delta(N-(k+i)) \right\rangle$$

i=0,1,…, $q+q_j-1$; $j \in \mathbb{R}$,

Step 1: Construct the finite Jordan pairs $\left(C_j \in \mathbb{R}^{r\times n_j}, J_j \in \mathbb{R}^{n_j\times n_j}\right)$ corresponding to the forward behavior of the system and let

$$C := \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix} \in \mathbb{R}^{r\times n}$$

$$J := blockdiag\begin{bmatrix} J_1 & J_2 & \cdots & J_k \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Step 2: Construct the infinite Jordan pairs $\left(C_k^D(k) \in \mathbb{R}^{r\times(q+q_j)} \quad J_k^D(k) \in \mathbb{R}^{(q+q_j)\times(q+q_j)}\right)$ corresponding to the backward behavior and let

$$C_\infty^D(k) := \begin{bmatrix} C_k^D(k) & C_{k+1}^D(k) & \cdots & C_r^D(k) \end{bmatrix} \in \mathbb{R}^{r\times\mu}$$

$$J_\infty^D(k) := blockdiag\begin{bmatrix} J_k^D(k) & J_{k+1}^D(k) & \cdots & J_r^D(k) \end{bmatrix} \in \mathbb{R}^{\mu\times\mu}$$

Step 3: Find the least q such that

$$S_{q-1}=\begin{pmatrix} C & C_\infty J_\infty{}^{q-1} \\ CJ & C_\infty J_\infty{}^{q-2} \\ \vdots & \vdots \\ CJ^{q-1} & C_\infty \end{pmatrix}$$

has full column rank.

Step 4: Find V such that $\begin{pmatrix} S_{q-2} \\ V \end{pmatrix}$ has full column rank

Step 5: Find

$$P=\begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} S_{q-1}^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} S_{q-2}$$

$$\begin{pmatrix} U_0 & U_1 & \cdots & U_{q-1} \end{pmatrix}=S_{q-1}^{-1}$$

Step 6: Define the polynomial matrix A(σ) as

$$A(\sigma)=V\left(I-P\right)\begin{pmatrix} \sigma I-\mathrm{T}_1 & 0 \\ 0 & \sigma T_2-I \end{pmatrix}\left(U_0+U_1\sigma+...+U_{q-1}\sigma^{q-1}\right)$$

The vector space B_F^D and B_B^D belong to the solution space of the system A(σ)β(k)=0. It must be noted though that in general, they do not span the whole behavior space of A(σ)β(k)=0. □

Example 39: We want to find an AR-representation A(σ)β(k)=0 with the following forward and backward behavior

$$\beta_1(k)=\underset{\beta_{1,1}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}2^k+\underset{\beta'_{1,0}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}k2^k=\underset{\beta_{1,1}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}2^k+\underset{\beta_{1,0}}{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}k2^{k-1} \text{ and}$$

$$\left(\beta^\infty(k)\right)=\beta_2(k)=\underset{x_{1,2}}{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}\delta(\mathrm{N}-k)+\underset{x_{1,1}}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}\delta(N-k+1)+\underset{x_{1,0}}{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}\delta(N-k+2)$$

Step 1: Create the Finite Jordan Pair

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \text{ and } J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Step 2: Now create the Infinite Jordan Pair

$$C_2 = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \ J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 3: Now let us find the desired q! We begin by setting q=1

The matrix $S_0 = \begin{pmatrix} C_1 & C_2 \end{pmatrix} = \left(\begin{array}{cc|ccc} 2 & 1 & 2 & -1 & -1 \\ 0 & -1 & 0 & 1 & -1 \end{array}\right)$

does not have full column rank

For q=2 the matrix $S_1 = \begin{pmatrix} C_1 & C_2 J_2 \\ C_1 J_1 & C_2 \end{pmatrix} = \left(\begin{array}{cc|ccc} 2 & 1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ \hline 4 & 4 & 2 & -1 & -1 \\ 0 & -2 & 0 & 1 & -1 \end{array}\right)$

does not have full column rank.

For q=3 the matrix $S_2 = \begin{pmatrix} C_1 & C_2 J_2^2 \\ C_1 J_1 & C_2 J_2 \\ C_1 J_1^2 & C_2 \end{pmatrix} = \left(\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 4 & 4 & 0 & 2 & -1 \\ 0 & -2 & 0 & 0 & 1 \\ \hline 8 & 12 & 2 & -1 & -1 \\ 0 & -4 & 0 & 1 & -1 \end{array}\right)$

has Rank=5. Thus q=3 is our choice.

Step 4: We must find matrix V such that $\begin{pmatrix} S_1 \\ V \end{pmatrix}$ has full column rank!

For V= $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

we have that $\begin{pmatrix} S_1 \\ V \end{pmatrix} = \left(\begin{array}{ccccc} 2 & 1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 4 & 4 & 2 & -1 & -1 \\ 0 & -2 & 0 & 1 & -1 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right)$

has full column rank!

Step 5: $P = \begin{pmatrix} I_2 & 0 \\ 0 & J_2 \end{pmatrix} S_2^{-1} \begin{pmatrix} I_4 \\ 0_2 \end{pmatrix} S_1 = \begin{pmatrix} 1 & \frac{24}{17} & -\frac{6}{17} & \frac{6}{17} & \frac{45}{17} \\ 0 & \frac{1}{17} & \frac{4}{17} & -\frac{4}{17} & -\frac{13}{17} \\ 0 & -\frac{24}{17} & \frac{23}{17} & -\frac{6}{17} & -\frac{96}{17} \\ 0 & -\frac{20}{17} & \frac{5}{17} & \frac{12}{17} & -\frac{46}{17} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Step 6: $A(\sigma) = V\left(I_5 - P\right)\begin{pmatrix} \sigma I_2 - J_1 & 0 \\ 0 & \sigma J_2 - I_3 \end{pmatrix}\left(U_0 + U_1\sigma + U_2\sigma^2\right) =$

$$\begin{pmatrix} -\frac{10}{17}(s-2) & \frac{2}{17}\left(48-49s+10s^2\right) \\ \frac{1}{578}\left(716-628s+135s^2\right) & \frac{1}{578}\left(3192-2846s+1121s^2-270s^3\right) \end{pmatrix}$$

Taking the determinant of A(σ) we can check that

$$\text{Det(A(σ))}= \frac{18}{289}\left(s-2\right)^2\left(15s-2\right)$$

So the system A(σ) does actually have a finite elementary divisors at s=2 of order 2 as we wanted! (it also has more zeros, so our given behavior does not span the whole solution space)

Now we want to check if it also has an infinite elementary divisor at infinity of order 3! To check this we will take a look at the dual matrix of A(σ).

$$\tilde{A}(\sigma) =$$

$$\begin{pmatrix} -\frac{10}{17}(s^2 - 2s^2) & \frac{2}{17}\left(48s^3 - 49s^2 + 10s\right) \\ \frac{1}{578}\left(716s^3 - 628s^2 + 135s\right) & \frac{1}{578}\left(3192s^3 - 2846s^2 + 1121s - 270\right) \end{pmatrix}$$

with Det= $-\frac{18}{289}s^3\left(-15+2s\right)\left(2s-1\right)^2$

So the dual matrix has a Smith form at zero of order 3

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

and the matrix A(σ) has a Smith form at infinity

$$S^\infty_{A(\sigma)}(\sigma) = \sigma^3 S^0_{\tilde{A}(\sigma)}(\frac{1}{\sigma}) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & 1 \end{pmatrix}$$

This agrees with our theoretical results because we know from theorem 23 that the vector $\beta_j^\infty(k)$ is the sum of $q + q_j$ vectors. In this example, we have that q=3 and $q + q_j = 3$. This means that $q_j = 0$. This result agrees with the Smith form of A(σ) at infinity and the Smith form of the dual matrix at zero.

Finally, we check the properties regarding the Jordan Pairs $\begin{pmatrix} C_1 & J_1 \end{pmatrix}$ and $\begin{pmatrix} C_2 & J_2 \end{pmatrix}$

$$A_0C_1 + A_1C_1J_1 + A_2C_1J_1{}^2 + A_3C_1J_1{}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; rank\begin{pmatrix} C_1 \\ C_1J_1 \end{pmatrix} = 2$$

$$A_3C_2 + A_2C_2J_2 + A_1C_2J_2{}^2 + A_0C_2J_2{}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; rank\begin{pmatrix} C_2 \\ C_2J_2 \\ C_2J_2{}^2 \end{pmatrix} = 3$$

□

Example 40: We want to find an AR-representation A(σ)β(k)=0 with the following forward and backward behavior

$$\beta_1(k)=\underset{\beta_{1,1}}{\begin{pmatrix}1\\0\end{pmatrix}}+\underset{\beta_{1,0}}{\begin{pmatrix}1\\-1\end{pmatrix}}k \text{ and } \beta_2(k)=\underset{x_{1,1}}{\begin{pmatrix}-1\\-1\end{pmatrix}}\delta(\mathrm{N}-k)+\underset{x_{1,0}}{\begin{pmatrix}-1\\1\end{pmatrix}}\delta(N-k+1)$$

Step 1: We define the finite Jordan pair pair

$$C_1=\begin{pmatrix}\beta_{1,0} & \beta_{1,1}\end{pmatrix}=\begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix} \text{ and } J_1=\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

Step 2: The infinite Jordan pair is

$$\mathrm{C}_2=\begin{pmatrix}-1 & -1\\1 & -1\end{pmatrix};\mathrm{J}_2=\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

Step 3: Now let us find the desired q! We begin by setting q=1

The matrix $S_0=\begin{pmatrix}C_1 & C_2\end{pmatrix}$ is neither invertible, nor has full column rank.

For q=2 the matrix $S_1=\begin{pmatrix}C_1 & C_2J_2\\C_1J_1 & C_2\end{pmatrix}=\left(\begin{array}{cc|cc}1 & 1 & 0 & -1\\-1 & 0 & 0 & 1\\\hline 1 & 2 & -1 & -1\\-1 & -1 & 1 & -1\end{array}\right)$

has full column rank.

Step 4: We must find matrix V such that $\begin{pmatrix}S_0\\V\end{pmatrix}$ has full column rank!

For V= $\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\end{pmatrix}$

$$\begin{pmatrix}S_1\\V\end{pmatrix}=\left(\begin{array}{cccc}1 & 1 & -1 & -1\\-1 & 0 & 1 & -1\\\hline 1 & 0 & 0 & 1\\0 & 1 & 1 & 0\end{array}\right)$$

This matrix has full column rank.

Step 5: $P=\begin{pmatrix} I_2 & 0 \\ 0 & J_2 \end{pmatrix} S_1^{-1} \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix} S_0 = \begin{pmatrix} 1 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Step 6: $A(\sigma)=V\left(I_4-P\right)\begin{pmatrix} \sigma I_2 - J_1 & 0 \\ 0 & \sigma J_2 - I_2 \end{pmatrix}\left(U_0+U_1\sigma\right)=$

$$\begin{pmatrix} \frac{1}{2}\left(-4+3s-s^2\right) & \frac{1}{2}\left(-2+s-s^2\right) \\ \frac{1}{2}\left(-6+5s-s^2\right) & \frac{1}{2}\left(-4+3s-s^2\right) \end{pmatrix}$$

Taking the Determinant of A(σ) we can easily check that

$$Det\left(A(s)\right)=(s-1)^2$$

So the system A(σ) does actually have a finite elementary divisor at σ=1 of order 2!

For the infinite elementary divisor we need to take a look at the dual matrix

$$\tilde{A}(\sigma)=$$

$$\begin{pmatrix} \frac{1}{2}\left(-4s^2+3s-1\right) & \frac{1}{2}\left(-2s^2+s-1\right) \\ \frac{1}{2}\left(-6s^2+5s-1\right) & \frac{1}{2}\left(-4s^2+3s-1\right) \end{pmatrix}$$

with a determinant of

$$Det=(s-1)^2 s^2$$

So the dual matrix has a Smith form at zero

$$S^0_{\tilde{A}(\sigma)}(\sigma)=\begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

and matrix A(σ) has a Smith form at infinity

$$S^{\infty}_{A(\sigma)}(\sigma) = \sigma^2 S^0_{\tilde{A}(\sigma)}(\frac{1}{\sigma}) = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Finally, we check the properties regarding the Jordan Pairs $\begin{pmatrix} C_1 & J_1 \end{pmatrix}$ and $\begin{pmatrix} C_2 & J_2 \end{pmatrix}$

$$A_0C_1 + A_1C_1J_1 + A_2C_1J_1{}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ rank\begin{pmatrix} C_1 \\ C_1J_1 \end{pmatrix} = 2$$

$$A_3C_2 + A_2C_2J_2 + A_1C_2J_2{}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ rank\begin{pmatrix} C_2 \\ C_2J_2 \end{pmatrix} = 2$$

□

Example 41: We want to find an AR-representation A(σ)β(k)=0 with the following forward and backward behavior

$$\beta_1(k) = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\beta_{1,1}} 2^k + \underbrace{\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}}_{\beta'_{1,0}} k2^k = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\beta_{1,1}} 2^k + \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\beta_{1,0}} k2^{k-1}$$

$$\beta_2(k) = \underbrace{\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}}_{x_{1,2}} \delta(\mathrm{N}-k) + \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{x_{1,1}} \delta(N-k+1) + \underbrace{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}}_{x_{1,0}} \delta(N-k+2)$$

Step 1: Create the finite Jordan Pair

$$C_1 = \begin{pmatrix} \beta_{1,0} & \beta_{1,1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}; \ J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Step 2: Create the infinite Jordan pair

$$C_\infty = \begin{pmatrix} x_{1,0} & x_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}; \; J_\infty = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 3: We will now start assuming values for q. For q=1 the matrix

$$S_0 = \begin{pmatrix} C_1 & C_\infty \end{pmatrix} = \left(\begin{array}{cc|ccc} 2 & 1 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 \end{array}\right)$$ does not have full column rank.

For q=2 the matrix $S_1 = \begin{pmatrix} C_1 & C_\infty J_\infty \\ C_1 J_1 & C_\infty \end{pmatrix} = \left(\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & -2 \\ \hline 4 & 4 & 0 & 1 & -1 \\ 0 & -2 & 2 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{array}\right)$

has rank=5, thus it has full column rank, so q=2 is the accepted value.

Step 4: We must find a matrix V, such that the matrix $\begin{pmatrix} S_0 \\ V \end{pmatrix}$ has full column rank.

For

$$V = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

we have that

$$\begin{pmatrix} S_0 \\ V \end{pmatrix} = \left(\begin{array}{ccccc} 2 & 1 & 0 & 1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

has full column rank.

Step 5: $$P = \begin{pmatrix} I_2 & 0 \\ 0 & J_\infty \end{pmatrix} S_1^{-1} \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix} S_0 = \begin{pmatrix} \frac{-17}{23} & \frac{-40}{23} & \frac{63}{23} & \frac{-33}{23} & \frac{-72}{23} \\ \frac{44}{23} & \frac{67}{23} & \frac{-90}{23} & \frac{57}{23} & \frac{93}{23} \\ \frac{24}{23} & \frac{24}{23} & \frac{-24}{23} & \frac{29}{23} & \frac{34}{23} \\ \frac{24}{23} & \frac{24}{23} & \frac{-24}{23} & \frac{29}{23} & \frac{34}{23} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 6: $$A(\sigma) = V\left(I_5 - P\right)\begin{pmatrix} \sigma I_2 - J_1 & 0 \\ 0 & \sigma J_\infty - I_3 \end{pmatrix}\left(U_0 + U_1\sigma\right) =$$

$$\begin{pmatrix} \frac{56}{529}\left(3-5s+2s^2\right) & \frac{-1643}{529}+s+\frac{56}{529}s^2 & \frac{1}{529}\left(-946+305s+112s^2\right) \\ \frac{1}{23}\left(-9+48s-28s^2\right) & -\frac{2}{23}\left(-105+40s+7s^2\right) & \frac{1}{23}\left(-132-35s-28s^2\right) \\ \frac{1}{529}\left(237+732s+112s^2\right) & \frac{1}{529}\left(2037+506s+56s^2\right) & \frac{2}{529}\left(1137+394s+56s^2\right) \end{pmatrix}$$

with a determinant of

$$Det\left[A(s)\right] = \frac{33}{529}(s-2)^2(9+14s)$$

Now we will check if the dual matrix has a finite elementary divisor at zero of order 3.

$$\tilde{A}(\sigma) =$$

$$\begin{pmatrix} \frac{56}{529}\left(3s^2-5s+2\right) & \frac{-1643}{529}s^2+s+\frac{56}{529} & \frac{1}{529}\left(-946s^2+305s+112\right) \\ \frac{1}{23}\left(-9s^2+48s-28\right) & -\frac{2}{23}\left(-105s^2+40s+7\right) & \frac{1}{23}\left(-132s^2-35s-28\right) \\ \frac{1}{529}\left(237s^2+732s+112\right) & \frac{1}{529}\left(2037s^2+506s+56\right) & \frac{2}{529}\left(1137s^2+394s+56\right) \end{pmatrix}$$

with a determinant of

$$Det\left[\tilde{A}(s)\right]=\frac{33}{529}s^3(2s-1)^2(9s+14)$$

so the Smith form at zero of the dual matrix is

$$S^0_{\tilde{A}(\sigma)}(\sigma)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$$

which means that the Smith for at infinity of A(σ) is

$$S^\infty_{A(\sigma)}(\sigma)=\sigma^2 S^0_{\tilde{A}(\sigma)}(\frac{1}{\sigma})=\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \frac{1}{\sigma} \end{pmatrix}$$

so the system A(σ) actually has a zero at infinity of order 1, which agrees with our theoretical results because we know from theorem 23 that the vector $\beta_j^\infty(k)$ is the sum of $q+q_j$ vectors. In this example, we have that q=2 and $q+q_j=3$. This means that $q_j=1$. This result agrees with the Smith form of A(σ) at infinity and the Smith form of the dual matrix at zero.

We can also check in Mathematica that the matrix pairs $\left(C_1 \quad J_1\right)$ and $\left(C_\infty \quad J_\infty\right)$ are indeed the finite and infinite Jordan pairs of the matrix we found, because they satisfy

$$A_2C_1J_1^{\,2}+A_1C_1J_1+A_0C_1=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};\quad rank\begin{pmatrix} C_1 \\ C_1J_1 \end{pmatrix}=2$$

$$A_0 C_\infty J_\infty{}^2 + A_1 C_\infty J_\infty + A_2 C_\infty = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad rank \begin{pmatrix} C_\infty \\ C_\infty J_\infty \\ C_\infty J_\infty^2 \end{pmatrix} = 3$$

Second approach:

In the first method, we started by knowing from the form of $\beta_2(k)$ that $q + q_1 = 3$. We first assumed that q=1, hence $q_1 = 2$, but the matrix S_0 did not have full column rank, so we continiued assuming that q=2 (so $q_1 = 1$). For this case the algorithm worked fine and we got the desired result. The Smith form of the dual system at zero was

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} \sigma^{q-q} & 0 & 0 \\ 0 & \sigma^{q-q} & 0 \\ 0 & 0 & \sigma^{q+q_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$$

Although this method for q=2 worked fine and is absolutely theoretically correct, the truth is that we can examine other possible cases.

What we could assume, is that there is also another i.e.d. at infinity, which means another f.e.d. of the dual system at zero, of order $q' \le q_1 = 2$. The possible values for q' are $q' = 1$ $q' = 2$ or $q' = -1$ (in case we have a pole at infinity). This means that the Smith form of the dual system at zero could be

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} \sigma^{q-q} & 0 & 0 \\ 0 & \sigma^{q+q'} & 0 \\ 0 & 0 & \sigma^{q+q_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$$

or

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} \sigma^{q-q} & 0 & 0 \\ 0 & \sigma^{q+q'} & 0 \\ 0 & 0 & \sigma^{q+q_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma^3 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$$

or

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \begin{pmatrix} \sigma^{q-q} & 0 & 0 \\ 0 & \sigma^{q-q'} & 0 \\ 0 & 0 & \sigma^{q+q_1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$$

We will study the first case, where there is an extra i.e.d. of order 1.

To this i.e.d we correspond an infinite Jordan Pair (q+q'=2)

$$C_3 = \begin{pmatrix} \alpha & \delta \\ \beta & \varepsilon \\ \gamma & \zeta \end{pmatrix} \; ; \; J_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The combined Infinite Spectral Pair will be

$$C_\infty = \begin{pmatrix} C_2 & C_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & \alpha\,\delta \\ 2 & 0 & -1 & \beta\,\varepsilon \\ -1 & -1 & 0 & \gamma\,\zeta \end{pmatrix} \; ; \; J_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We will now continue with the algorithm

Step 3: For q=2, we must find α,β,γ,δ,ε,ζ such that the matrix

$$S_1 = \begin{pmatrix} C_1 & C_\infty J_\infty \\ C_1 J_1 & C_\infty \end{pmatrix}$$

has full column rank. But this matrix is 6-by-7, so it cannot have full column rank. This means that we cannot solve this problem for q=2, so the case for q=3 has to be examined. But since we have already proved that this problem has a solution for q=2, by solving the problem for the case of q=3, we lose one of the most important aims we had, which is finding a system of the smallest possible degree q, satisfying a given behavior. The same thing goes for $q' = 2$.

So the two cases of adding a zero at infinity of our choise are discarded, because they will lead to a system of higher degree.

Let's now consider the case where there is an extra pole at infinity. in this case, To this i.e.d. we correspond an infinite Jordan Pair

$$C_3 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \; ; \; J_3 = 0$$

The combined Infinite Spectral Pair will be

$$C_\infty = \begin{pmatrix} C_2 & C_3 \end{pmatrix} = \left(\begin{array}{ccc|c} 0 & 1 & -1 & \alpha \\ 2 & 0 & -1 & \beta \\ -1 & -1 & 0 & \gamma \end{array}\right) \; ; \; J_\infty = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We will now continue with the algorithm

Step 3: The case of q=1 needs not to be examined, we can be sure that S_0 will not have full column rank. For q=2, we must find α,β,γ such that the matrix

$$S_1 = \begin{pmatrix} C_1 & C_\infty J_\infty \\ C_1 J_1 & C_\infty \end{pmatrix} = \left(\begin{array}{cc|cccc} 2 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ \hline 4 & 4 & 0 & 1 & -1 & \alpha \\ 0 & -2 & 2 & 0 & -1 & \beta \\ -2 & -1 & -1 & -1 & 0 & \gamma \end{array}\right)$$

has full column rank. For α=β=1 and γ=0 this criterion is satisfied. So q=2.

Step 4: We must find a matrix V, such that the matrix $\begin{pmatrix} S_0 \\ V \end{pmatrix}$ has full column rank.

For

$$V = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

the matrix

$$\begin{pmatrix} S_0 \\ V \end{pmatrix} = \left(\begin{array}{cccccc} 2 & 1 & 0 & 1 & -1 & 1 \\ 0 & -1 & 2 & 0 & -1 & 1 \\ -1 & 0 & -1 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

has full column rank.

Step 5: $P=\begin{pmatrix} I_2 & 0 \\ 0 & J_\infty \end{pmatrix} S_1^{-1} \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix} S_0 = \begin{pmatrix} \frac{5}{13} & \frac{-8}{13} & \frac{21}{13} & \frac{6}{13} & \frac{-6}{13} & \frac{6}{13} \\ \frac{4}{13} & \frac{17}{13} & \frac{-30}{13} & \frac{-3}{13} & \frac{3}{13} & \frac{-3}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{13} & \frac{8}{13} & \frac{-8}{13} & \frac{7}{13} & \frac{6}{13} & \frac{-6}{13} \\ \frac{-4}{13} & \frac{-4}{13} & \frac{4}{13} & \frac{3}{13} & \frac{10}{13} & \frac{-10}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Step 6: $A(\sigma)=V\left(I_6-P\right)\begin{pmatrix} \sigma I_2 - J_1 & 0_{2,4} \\ 0_{4,2} & \sigma J_\infty - I_4 \end{pmatrix}\left(U_0+U_1\sigma\right)=$

$$A(\sigma)=\begin{pmatrix} \frac{2}{13}\left(5-4\sigma+\sigma^2\right) & \frac{1}{13}\left(-5+3\sigma+\sigma^2\right) & \frac{1}{13}\left(18-11\sigma+2\sigma^2\right) \\ \frac{-2}{13}\left(1-8\sigma+3\sigma^2\right) & \frac{1}{13}\left(27-13\sigma-3\sigma^2\right) & \frac{1}{13}\left(-14+25\sigma-6\sigma^2\right) \\ \frac{1}{13}\left(-17-5\sigma+6\sigma^2\right) & \frac{1}{13}\left(15-\sigma+3\sigma^2\right) & \frac{2}{13}\left(-1-7\sigma+3\sigma^2\right) \end{pmatrix}$$

with a determinant of $Det\left[\mathrm{A}(\sigma)\right]=\frac{12}{13}(-2+s)^2$

The dual system is

$$\tilde{A}(\sigma)=\begin{pmatrix} \frac{2}{13}\left(5\sigma^2-4\sigma+1\right) & \frac{1}{13}\left(-5\sigma^2+3\sigma+1\right) & \frac{1}{13}\left(18\sigma^2-11\sigma+2\right) \\ \frac{-2}{13}\left(1\sigma^2-8\sigma+3\right) & \frac{1}{13}\left(27\sigma^2-13\sigma-3\right) & \frac{1}{13}\left(-14\sigma^2+25\sigma-6\right) \\ \frac{1}{13}\left(-17\sigma^2-5\sigma+6\right) & \frac{1}{13}\left(15\sigma^2-\sigma+3\right) & \frac{2}{13}\left(-1\sigma^2-7\sigma+3\right) \end{pmatrix}$$

with a determinant of $Det\left[\tilde{\mathrm{A}}(\sigma)\right]=\frac{12}{13}\sigma^4(-1+2\sigma)^2$ and

$$S^0_{\tilde{A}(\sigma)}(\sigma)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix}$$

We can also check that

$$A_2C_1J_1^{\ 2} + A_1C_1J_1 + A_0C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad rank\begin{pmatrix} C_1 \\ C_1J_1 \end{pmatrix} = 2$$

$$A_0C_\infty J_\infty^{\ 2} + A_1C_\infty J_\infty + A_2C_\infty = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad rank\begin{pmatrix} C_\infty \\ C_\infty J_\infty \\ C_\infty J_\infty^2 \end{pmatrix} = 4$$

CONCLUSIONS

Overall, we have studied the behavior of discrete-time AR-representations, the solution space of such a system and the construction of a square matrix A(σ) such that the corresponding AR-representation satisfies a given behavior. More specifically, we managed to give a theorem connecting the backward behavior of a system to the forward beahvior of it's dual representation corresponding to the f.e.d. at zero. In addition, we provided two methods of constructing a system with a given forward and backward behavior. The first algorithm was implemented to mathematica.

In the algorithms presented by Gohberg, it is assumed that the whole spectral information will be given. We have seen that even when this is not the case, our algorithms still function, but with complications (e.g. extra behavior).

These methods where studied for square systems of difference equations, yet it is possible that these results can be implemented to the case of non-square systems.

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